XVII. *Memoir on* ABEL'S *Theorem.*

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Received May 27,—Read June 10, 1880.

THE object of this paper is to present in a shortened and simplified form the processes and the results of ABEL'S famous memoir 'Sur une propriété générale d'une classe très-étendue de fonctions transcendantes,' composed and offered to the French Institute in 1826, but first published in the 'Mémoires des Savans Étrangers' for 1841 (pp. 176–264).

The generality and the power of this memoir are well known, but its form is not attractive. BOOLE indeed in a paper on a kindred subject (Phil. Trans. for 1857, pp. 745–803) says: "As presented in the writings of ABEL . . . the doctrine of the comparison of transcendants is repulsive, from the complexity of the formulæ in which its general conclusions are embodied." BOOLE'S theorems however escape this charge only with loss of the generality which makes ABEL'S valuable.

But this complexity is rather apparent than fundamental. It is here attempted, by re-arrangement of parts, by separation of essential from non-essential steps, by changes of notation, in particular by the introduction of a symbol and a theorem discussed by BOOLE in the paper already referred to and by the addition of examples of the processes and results, to reduce this part of an important subject to a shape more simple, while no less general, than the original.

The first of the three sections into which the following paper is divided contains (arts. 1–10) the investigation of the principal theorem of ABEL'S memoir: these articles correspond to pp. 176–196 of the original, but are much simplified by the aid of BOOLE'S proposition: the theorem is written at the end of art. 8 in the form

$$\Sigma\int f(x, y)dx=\Theta\left[\frac{1}{f_2(x)\mathrm{F}_0(x)}\right]\mathrm{F}_0(x)\Sigma\frac{f_1(x, y)}{\chi'(y)}\log\theta(y)+\mathrm{C}$$

and answers to ABEL'S equation (37), p. 193.

In art. 11 three examples are given of ABEL'S theorem. Those have been chosen of which the results were well known (*e.g.*, the circular and elliptic functions) with a view to the comparison of this and less general methods.*

* For other methods of solution compare LESLIE ELLIS, B.A. Reports for 1846, p. 38; LEGENDRE, 'Fonc. Ell.,' t. iii., p. 192; BOOLE, *loc. cit.*, arts. 18, 24.

In the second section (arts. 12–20) it is shown to follow from the results of the first that the sum of any number of integrals of the form considered may be expressed in terms of a definite number of such integrals, and the question what is the least value of this definite number is discussed: the result is stated at the end of art. 20. These articles correspond to pp. 211–228 in the original; they are rendered more direct by the nomenclature of 'major terms' and 'sets,' the introduction of the letter τ, and various minor changes of notation.

Art. 21 contains an example of the method of this section.

The third section contains two distinct parts: first, a generalization (art. 22) of the theorem of Section I., showing that a similar expression to that obtained there may be found for the sum of any number of such integrals each multiplied by any rational number positive or negative, integral or fractional; secondly, an investigation (art. 23) of the conditions necessary that the algebraic expression obtained for the sum of the integrals considered in Section I.—*i.e.*, the right-hand member in the main theorem—may reduce to a constant. This article corresponds to pp. 196–208 in ABEL, but the demonstration is greatly shortened and simplified by its being placed after (instead of, with ABEL, before) Section II.

ABEL concludes by applying his methods to the case of integrals of the form

$$\int \frac{\phi}{\psi^{\frac{m}{n}}} dx.$$

I have succeeded in shortening the necessary work, but my process and result are so similar to those of the original as hardly to be worth reproducing here.

An appendix contains an algebraical lemma and a list—it is hoped complete—of the errata in the original memoir. It appeared to the writer worth while to attempt to save subsequent readers the considerable inconvenience these errata had caused himself.

There follows an addition from Professor CAYLEY, wherein it is shown that the expression found in art. 20 for the least value of the number of conditions connecting the variables of the integrals we sum is equal to the deficiency (*Geschlecht*) of the curve represented by the equation $\chi(x, y)=0$. That this least value is equal to the deficiency is a leading result in RIEMANN's theory of the Abelian integrals; the assumptions made in the text as to the form of the roots of the equation $\chi(x, y)=0$ considered as an equation for the determination of y are equivalent to the assumption that the curve $\chi(x, y)=0$ has certain singularities; and it is in the addition shown that the resulting value of the deficiency, as calculated by the formulæ in Professor CAYLEY's paper 'On the Higher Singularities of a Plane Curve,' Quart. Math. Journ., vol. vii. (1866), pp. 212–222, has in fact the foregoing value.

SECTION I.

1. The general question to which an answer is sought in what is called the Theory of the Comparison of Transcendants may be stated thus :—

Is it always possible to establish, between the values for different variables of the integral of an algebraic function however complex, algebraic relations: the variables themselves being connected by any requisite algebraic laws?

If, for example,

$$\int X dx = F(x)$$

where X is any algebraic function of x, rational or irrational, integral or fractional, is it necessarily possible by connecting $x_1, x_2, \ldots x_n$ by any requisite algebraic laws to obtain an algebraic (or logarithmic) expression for the sum

$$F(x_1)+F(x_2)+\ldots+F(x_n)\,?$$

This question is suggested on the one hand by such well-known results as

$$F(x_1)+F(x_2)=\text{constant, where } X=\frac{1}{\sqrt{1-x^2}}, \text{ if } x_1^2+x_2^2=1$$

and

$$F(x_1)+F(x_2)+F(x_3)=0 \quad \text{where } X=\frac{1}{\sqrt{1-x^2.1-k^2x^2}}$$

if

$$4(1-x_1^2)(1-x_2^2)(1-x_3^2)=(2-x_1^2-x_2^2-x_3^2+k^2x_1^2x_2^2x_3^2)^2,$$

and on the other hand by the possibility of finding algebraical expressions for many symmetric functions of the roots of equations though these roots may not separately be so expressible.

It is in fact this combination of the theory of integrals and the theory of equations which furnishes the key to the problem; enabling us to express the requisite algebraical laws very concisely by a *single* equation of which the variables are roots, and whose coefficients are not independent but connected by a corresponding number of relations.

2. *The expression of the function to be integrated.*

To escape the inconvenience of fractional and irrational forms we first introduce two new functions and a fresh variable.

Whatever be the nature of the function X—the subject of integration in the transcendants we are considering—it may be written

$$f(x, y)$$

where f is a rational (but not necessarily integral) function of x and y, while y is determined as a function of x from the equation

$$\chi(y) \equiv y^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots + p_1 y + p_0 = 0$$

the p's being rational integral functions of x.

This is clear since any explicit irrational function is the root of an equation with integral and rational coefficients, in which, by a suitable change of variable, the highest coefficient can be made unity.

4. The shape in which it is most convenient to deal with $f(x, y)$, and in which we shall in future assume it to be expressed, is obtained when its denominator is made the product of $\chi'(y)$—the differential coefficient of $\chi(y)$ with respect to y—and a function of x only.

This can always be done; for let

$$\begin{aligned} f(x, y) &= \frac{F_1(x, y)}{F_2(x, y)} \\ &= \frac{F_1(x, y)\chi'(y)}{F_2(x, y)\chi'(y)} \\ &= \frac{F_1(x, y)\chi'(y)F_2(x, y_2)F_2(x, y_3)\ldots F_2(x, y_n)}{\chi'(y).F_2(x, y_1)F_2(x, y_2)F_2(x, y_3)\ldots F_2(x, y_n)} \end{aligned}$$

$y_1, y_2 \ldots y_n$ being the n roots of the equation

$$\chi = 0$$

and therefore functions of x; and y_1 being the root which we have before denoted by y.

Now the product $F_2(x, y_1) \ldots F_2(x, y_n)$ involving only symmetrical functions of the y's may be expressed as a function of x only; while, using the equations

$$\sum_{r=2}^{r=n} y_r = \sum_{r=1}^{r=n} y_r - y_1$$

$$\sum_{r=2}^{r=n}\sum_{s=2}^{s=n} y_r y_s = \sum_{r=1}^{r=n}\sum_{s=1}^{s=n} y_r y_s - y_1 \sum_{r=2}^{r=n} y_r$$

$$\&c. = \&c.$$

and, lastly,

$$y_2 y_3 \ldots y_n = \frac{(-)^n p_0}{y_1},$$

the product

$$F_2(x, y_2)F_2(x, y_3)\ldots F_2(x, y_n)$$

can be expressed as a rational function of x, y; while $F_1(x, y)$ and $\chi'(y)$ are rational and integral functions.

So $f(x, y)$ the subject of integration is reduced to the form

$$\frac{f_1(x, y)}{f_2(x)\chi'(y)}$$

in which it will hereafter be used.

5. *The equation whose roots are the variables of the functions we compare.*

This equation is clearly not arbitrary; for if it were we could choose it linear; and having then only a single integral, should be required to find for it an algebraic (or logarithmic) expression, a thing generally impossible.

We shall find it sufficient to take, for this equation, the result of eliminating y between χ and any other integral function of x, y; which, by the use of χ, can, of course, be made of (at most) the $(n-1)^{\text{th}}$ degree in y.

Let this second function be

$$\theta(y) \equiv q_{n-1}y^{n-1} + q_{n-2}y^{n-2} + \ldots + q_1 y + q_0$$

and let the result of elimination, viz.:—

$$\theta(y_1)\theta(y_2) \ldots \theta(y_n)$$

be denoted by E.

$\theta = 0$ may be called the *equation of condition.*

We assume $q_0, q_1 \ldots q_{n-1}$ to be rational integral functions of x; while any number of the coefficients in these functions are arbitrary: call them $a_1, a_2, \ldots$

E will then be a rational integral function of x and these quantities $a_1, a_2, \ldots$

We may then either (1) take the roots of the equation $E=0$,—$a_1, a_2, \ldots$ being considered absolute constants—as the upper limits of our integrals (of which alone we view these integrals as functions); or (2) since by a due alteration of the a's we may produce any possible simultaneous alteration of the x's, we may consider the variables x in the different integrals as, in the passage from the lower to the higher limit, always connected by the equation $E=0$, in which now $a_1, a_2, \ldots$ are a system of variables with which the variation of x has to be connected. The latter, as the more general and powerful hypothesis, is to be preferred.

$E=0$ may be called the *equation of the limits,* or the *equation of transformation.*

6. It may happen that, owing to a relation connecting the a's, the equation $E=0$ is satisfied by values of x independent of these new variables. This relation, since one of the θ's of which E is the product will vanish for this value of x and θ is linear in the a's, must be a linear relation. We will then suppose

$$E(x, a_1, a_2, \ldots) = F_0(x)F(x)$$

where $F_0(x)$ is independent of the a's; and, the degree of $F(x)$ being μ, let its roots be

$x_1, x_2, \ldots x_\mu$; let the corresponding values of y, the root of χ with which we are concerned, be $y_{11}, y_{12}, \ldots y_{1\mu}$.*

7. Having expressed $f(x, y)$ in a convenient shape we have next *to transform the* dx *of our integrals into the differentials of the new variables.*

If δ denotes the operation of differentiating with regard to our new variables we have from the equation $F=0$ by which x is connected with them

$$F'(x)dx+\delta F(x)=0$$

But

$$\delta E=F_0(x)\delta F(x)$$

therefore

$$dx=-\frac{\delta E}{F_0(x)F'(x)}$$

Again

$$E=\theta(y_1)\theta(y_2)\ldots\theta(y_n)$$

therefore

$$\delta E=\sum_{r=1}^{r=n}\frac{E}{\theta(y_r)}.\delta\theta(y_r)$$

* As an example of these processes let

$$X=\frac{1}{\sqrt{1+x^4}}$$

A natural assumption is

$$\chi(y)\equiv y^2-(1+x^4)=0$$

so that

$$f(x, y)=\frac{1}{y}.$$

Take for the second function the form

$$\theta(y)\equiv y-(1+a_1x+a_2x^2)$$

and on elimination we find

$$E(x, a_1, a_2)\equiv(a_2^2-1)x^3+2a_1a_2x^2+(a_1^2+2a_2)x+2a_1=0$$

Now, if we had

$$a_1+a_2=-1+\sqrt{2} \text{ (a } linear \text{ relation)}$$

$E=0$ would be satisfied on making $x=1$ and we should have

$$F_0(x)=x-1$$

while

$$F(x)=(a_2^2-1)x^2+(2a_1a_2+a_2-1)x+a_1^2+2a_1a_2+2a_2^2+a_2-1$$

and may be expressed in terms of a_1 alone.

We should also have

$$f(x, y)=\frac{1}{y}=\frac{2}{2y}=\frac{2}{\chi'(y)}$$

so that

$$f_1(x, y)=2, f_2(x)=1.$$

Now (using as before y or y_1 indifferently for the root with which we are concerned), we have $\theta(y_1)=0$: whence if $\lambda(x, y)$ be any rational function

$$\lambda(x, y)\delta E=\lambda(x, y)\sum_{r=1}^{r=n}\frac{E}{\theta(y_r)}.\delta\theta(y_r)$$

$$=\lambda(x, y_1)\frac{E}{\theta(y_1)}.\delta\theta(y_1)$$

all the other terms in Σ vanishing,

$$=\sum_{r=1}^{r=n}\lambda(x, y_r)\frac{E}{\theta(y_r)}.\delta\theta(y_r)$$

if we introduce a set of vanishing terms.

We have then obtained an expression for dx and a convenient modification of the result when the differential is multiplied by any function λ of x and y.

So, finally,

$$f(x, y)dx=-\frac{f_1(x, y)}{\chi'(y)f_2(x)}.\frac{\delta E}{F_0(x)F'(x)}$$

$$=-\frac{1}{f_2(x)F_0(x)F'(x)}\sum_{r=1}^{r=n}\frac{f_1(x, y_r)}{\chi'(y_r)}.\frac{E}{\theta(y_r)}\delta\theta(y_r).$$

8. From this point a symbol and theorem due to BOOLE* furnish a short path to the result. The symbol is thus defined :—

"If $\phi(x)f(x)$ be any function of x composed of two factors $\phi(x), f(x)$, whereof $\phi(x)$ is rational, let $\Theta[\phi(x)]f(x)$ denote the result obtained by successively developing the function in ascending powers of each simple factor $x-a$ in the denominator of $\phi(x)$, taking in each development the coefficient of $\frac{1}{x-a}$, adding together the coefficients thus obtained and subtracting from the result the coefficient of $\frac{1}{x}$ in the development of the same function $\phi(x)f(x)$ in descending powers of x."†

BOOLE's theorem is the following :—

"If $\phi(x)$ be any rational function of x and if $E=0$ be any equation, rational and integral with respect to x, by which x is connected with a new set of variables $a_1, a_2, \ldots$ then, provided that $\phi(x)$ does not become infinite when $E=0$, we have

$$\Sigma\phi(x)=-\Theta[\phi(x)]\frac{d\log E}{dx}\text{"}$$

the Σ indicating summation for the various roots of the equation $E=0$.

* Phil. Trans. for 1857, pp. 751, 757.

† CAUCHY employs in his 'Calcul des Résidus' a symbol $\mathcal{E}$ only differing from BOOLE's Θ by not including the subtractive term last mentioned. Any theorem can be instantly transferred from the one notation to the other.

Assuming the truth of this theorem we may proceed with the investigation as follows :—

Since $f_1(x, y)\frac{E}{\theta y}\delta\theta y$ is a rational integral function of x, y and may therefore be expressed in the form $\Sigma P_r y^r$, P_r being a rational integral function of x and r a positive integer not greater than $n-1$, while y is a root of the equation $\chi(y)=0$, we have, by a known theorem of partial fractions,*

$$\Sigma\frac{f_1(x, y)\frac{E}{\theta y}\delta\theta y}{\chi'(y)}=P_{n-1}$$

We have then, by art. 7,

$$\Sigma f(x, y)dx=\Sigma\left\{\frac{-P_{n-1}}{f_2(x)F_0(x)F'(x)}\right\}$$

$$=\Theta\left[\frac{P_{n-1}}{f_2(x)F_0(x)F'(x)}\right]\frac{d\log F}{dx}.$$

By BOOLE's theorem this

$$=\Theta\left[\frac{P_{n-1}}{f_2(x)F_0(x)F'(x)}\right]\frac{F'(x)}{F(x)}$$

$$=\Theta\left[\frac{1}{f_2(x)F_0(x)}\right]\frac{P_{n-1}}{F(x)}.$$

For since P_{n-1} is an integral function it contributes nothing to the interpretation of Θ by being within the square bracket : and, *if we assume that* $F'(x)$ *and* $F(x)$ *have no common factor* (which is also the case for $F'(x)$—which contains the a's—and $F_0(x)$ and $F_2(x)$—which do not), we shall have in the expansion of $\frac{P_{n-1}}{f_2(x)F_0(x)F'(x)}$ no term involving the reciprocal of a linear factor of $F'(x)$, which therefore may also be brought out of the square bracket.

The expression last obtained

$$=\Theta\left[\frac{1}{f_2(x)F_0(x)}\right]\frac{E}{F(x)}\Sigma\frac{f_1(x, y)}{\chi'(y)}\cdot\frac{\delta\theta y}{\theta y}$$

$$=\Theta\left[\frac{1}{f_2(x)F_0(x)}\right]F_0(x).\Sigma\frac{f_1(x, y)}{\chi'(y)}\cdot\frac{\delta\theta y}{\theta y}$$

Under this form the sum is immediately integrable, for the new variables (of which alone this is now a function) occur only in the factor $\frac{\delta\theta y}{\theta y}$.

Integrating we find

$$\Sigma\int f(x, y)dx=\Sigma\int\frac{f_1(x, y)}{f_2(x)\chi'(y)}dx=\Theta\left[\frac{1}{f_2(x)F_0(x)}\right]F_0(x)\Sigma\frac{f_1(x, y)}{\chi'(y)}\log\theta y+C.$$

* See also note on art. 10, (i.).

This is the general theorem for the summation of integrals of any form of which we were led to suspect the existence.

It corresponds to that numbered (37) on page 193 in ABEL'S Memoir (and which should be called "ABEL'S Theorem," though that name is frequently given to the very narrow case of it discussed on page 255), while it is more concise through the introduction of the symbol Θ, and more intelligible through the absence of the letter ν.*

9. In general, as has been said, the function E has no factor independent of the a's, *i.e.*, $F_0(x)=1$.

In this case the formula of the last article takes the simpler form

$$\Sigma\int f(x, y)dx=\Theta\left[\frac{1}{f_2(x)}\right]\Sigma\frac{f_1(x, y)}{\chi'(y)}\log \theta y+C$$

As an example of the expansion of Θ suppose $f_2(x)=(x-\alpha)^m$.

We have then

$$\text{coefficient of } \frac{1}{x-\alpha} \text{ in the expansion of } \frac{1}{(x-\alpha)^m}\Sigma\frac{f_1(x, y)}{\chi'(y)}\log \theta y$$

$$i.e., \text{ of } \frac{1}{(x-\alpha)^m}\Gamma(x), \text{ say,}$$

$$i.e., \text{ of } \frac{1}{(x-\alpha)^m}\{\Gamma(\alpha)+(x-\alpha)\Gamma'(\alpha)+\ldots\}$$

$$=\frac{1}{\underline{|m-1}}\Gamma^{m-1}(\alpha)$$

$$i.e., =\frac{1}{\underline{|m-1}}\cdot\overline{\frac{d}{d\alpha}}\Big|^{m-1}\left\{\Sigma\frac{f_1(\alpha, A)}{\chi'(A)}\log \theta A\right\}$$

where A is the value of y corresponding to $x=\alpha$; and—representing by $C_{\frac{1}{x}}\lambda(x)$ the coefficient of $\frac{1}{x}$ in the descending expansion of $\lambda(x)$—

$$\Sigma\int f(x, y)dx\equiv\Sigma\int\frac{f_1(x,y)}{(x-\alpha)^m\chi'(x)}dx$$

$$=\frac{1}{\underline{|m-1}}\overline{\frac{d}{d\alpha}}\Big|^{m-1}\left\{\Sigma\frac{f_1(\alpha, A)}{\chi'(\alpha)}\log \theta A\right\}-C_{\frac{1}{x}}\left\{\frac{f_1(x, y)}{(x-\alpha)^m\chi'(y)}\log \theta y\right\}+C,$$

which is identical with ABEL'S formula (44).

10. Before proceeding to examples of the use of the general theorem one or two points in the proof and the result should be alluded to.

(i.) *A limitation to the form of the function θ.*

In choosing this function we may not make $q_1=0$, $q_2=0$, ... $q_{n-1}=0$ simultaneously:

* The want of clearness spoken of is due to an ambiguity in the important sentence (p. 187) in which ABEL implicitly defines the letter ν which is to appear prominently in his enunciation of the final theorem. But it is hardly necessary to dwell on a difficulty which the method of the text avoids.

in other words, our function must not reduce itself so as to contain x only. This is clear *à priori;* for if it should so reduce itself we might choose for q_0 a linear function of x, which is generally impossible (art. 5).

It will be useful to examine at what point the assumption vitiates the subsequent demonstration. We should, in fact, have

$$\begin{aligned} \mathrm{E} &= \theta(y_1)\theta(y_2)\ \ldots\ \theta(y_n) \\ &= q_0 q_0 \ldots q_0 \\ &= q_0^{\,n} \end{aligned}$$

so that

$$\frac{\mathrm{E}}{\theta(y_r)} = q_0^{\,n-1}$$

and this vanishes for all the values of x obtained by putting $\mathrm{E}=0$, so that the right-hand side of the equation

$$f(x, y)dx = -\frac{1}{f_2(x)\mathrm{F}_0(x)\mathrm{F}'(x)}\Sigma\frac{f_1(x, y_r)}{\chi'(y_r)}\cdot\frac{\mathrm{E}}{\theta(y_r)}\delta\theta y_r$$

is identically zero, and the whole process invalid.*

* There is one case in which the function θ may be legitimately reduced to the single term q_0; viz. the case when χ is a linear function of y.

It is plain that, as $n=1$, we have not the difficulty of repeated roots which generally vitiates the result of this assumption.

In fact, let

$$\chi(y) = y + \alpha$$

while

$$\theta = a_0 + a_1 x + \ldots + x^n$$

Then

$$\mathrm{E} = \mathrm{F}(x) = \theta$$

and, as by p. 718,

$$dx = -\frac{\delta\mathrm{F}(x)}{\mathrm{F}'(x)} = -\frac{\delta\theta}{\theta'}$$

we have

$$\Sigma y dx = \Sigma\frac{\alpha}{\theta'}\cdot da_0 + \Sigma\frac{\alpha x}{\theta'}\cdot da_1 + \ldots + \Sigma\frac{\alpha x^{n-1}}{\theta'}\cdot da_{n-1}$$

As an example of which formula, let $\alpha = x^m$

so that

$$-\frac{\Sigma x^{m+1}}{m+1} = \int\Sigma\frac{x^m}{\theta'}da_0 + \int\Sigma\frac{x^{m+1}}{\theta'}da_1 + \ldots + \int\Sigma\frac{x^{m+n-1}}{\theta'}da_{n-1}$$

Put $m=0$ and we have

$$-\Sigma x = \int\Sigma\frac{1}{\theta'}da_0 + \int\Sigma\frac{x}{\theta'}da_1 + \ldots$$

But

$$-\Sigma x = a_{n-1}$$

whence

$$\Sigma\frac{x^k}{\theta'} = 0 \text{ if } k < n-1$$

while

$$\Sigma\frac{x^{n-1}}{\theta'} = 1$$

And this is the theorem (easy to prove otherwise) which was assumed in the course of the general demonstration on page 719.

(ii.) *The assumption* (in BOOLE's theorem) *that* ϕ *is not rendered infinite by the values which satisfy the equation* $E=0$; and *the assumption* (in art. 8) *that* $F(x)$ *and* $F'(x)$ *have no common factor.*

These assumptions are identical: for ϕ is rendered infinite by the vanishing of $f_2(x)F_0(x)F'(x)$, and, since the roots of F are all functions of the a's, they cannot satisfy the equations $f_2(x)=0$, $F_0(x)=0$, into which no a enters.

If then F and F′ have no common factor, the first assumption is justified.

We assert in this that $F=0$ is not an equation possessing equal roots—*i.e.*, that $x_1, x_2, \ldots x_\mu$ are all unequal. Suppose, on the contrary, that we have equal roots—say $x_1=x_2=x_3$.* If then y_1, y_2, y_3 are the corresponding roots of χ we shall have

$$\theta(y_1)=0, \quad \theta(y_2)=0, \quad \theta(y_3)=0$$

for the same value x_1 of x; and therefore in the expression of

$$-\frac{1}{f_2(x)F_0(x)F'(x)}\Sigma\frac{f_1(x,y_r)}{\chi'(y_r)}\cdot\frac{E}{\theta y_r}\delta\theta y_r$$

we have a term of the form $\frac{0}{0}$, viz.: that due to the root $x=x_1$, and it will be three times repeated.

We see then the character of the difficulty introduced by the equality of roots. It does not altogether vitiate the solution; it only requires that we should modify it by using, instead of the equations $\theta(y_1)=0$, $\theta(y_2)=0$, $\theta(y_3)=0$, the equations

$$\theta(y_1)=0, \quad \frac{d\theta y_1}{dx}=0, \quad \frac{d^2\theta y_1}{dx^2}=0.$$

The manner in which all the steps of the analysis and the final result are affected by this change is obvious.

11. It will now be natural to give examples of the application of the general theorem, and those are chosen the results of which are well-known, as furnishing comparison between this and other methods of research among transcendants. The second and third are treated by BOOLE, in the paper frequently referred to, as examples of his less general theorem.

I. *The function* $\sin^{-1} x$.

Let

$$X=\frac{1}{\sqrt{(1-x^2)}},$$

and take

$$\chi(y)=y^2+x^2-1,$$

* The reasoning will be applicable to any other number of equalities among the roots.

so that

$$f(x,\,y)=\frac{1}{y}\,;\ f_1(x,\,y)=2\,;\ f_2(x)=1.$$

Also let

$$\theta(y)=y+x-a.^{*}$$

Eliminating y we get

$$\mathrm{E}\equiv 2x^2-2ax+a^2-1=0$$

as the equation of the limits.

If x_1, x_2 are the roots of this equation we easily find

$$x_1^{\,2}+x_2^{\,2}=1.$$

The theorem then gives

$$\Sigma\int\frac{dx}{\sqrt{(1-x^2)}}=\Theta[1]\Sigma\frac{2}{y}\log\,(y-a+x)+\mathrm{C}$$

$$=-2\mathrm{C}_{\frac{1}{x}}\,\Sigma\frac{1}{y}\left(\log x+\frac{y-a}{x}-\frac{(y-a)^2}{2x^2}+\ldots\right)+\mathrm{C}.$$

But $\Sigma\frac{1}{y}=0$, wherefore the right-hand side reduces to a constant, and we have the result that

$$\int_0^{x_1}\frac{dx}{\sqrt{(1-x^2)}}+\int_0^{x_2}\frac{dx}{\sqrt{(1-x^2)}}$$

is constant if

$$x_1^{\,2}+x_2^{\,2}=1,$$

and so

$$=\int_0^1\frac{dx}{\sqrt{(1-x^2)}}\,;$$

and this is, of course, the well-known theorem that $\theta+\phi=\frac{\pi}{2}$ if $\sin^2\theta+\sin^2\phi=1$, (the angles being restricted to the first quadrant).

II. *The elliptic functions.*

As a second example take

$$\mathrm{X}=\frac{a+bx^2}{(1+nx^2)\sqrt{(1-x^2)(1-c^2x^2)}}$$

and let

$$\chi(y)\equiv y^2-(1-x^2)(1-c^2x^2)$$

so that

$$\int\mathrm{X}dx=\int\frac{2(a+bx^2)}{(1+nx^2)\chi'(y)}dx$$

and

$$f_1(x,\,y)=2(a+bx^2)$$
$$f_2(x)=1+nx^2.$$

* To choose the more general form $y+bx-a$ leads by similar steps to a less interesting result.

Also take

$$\theta(y) \equiv y-(1+px+qx^2),$$

so that removing the factor $x=0$ (see p. 732), we get by elimination of y between χ and θ

$$E \equiv (q^2-c^2)x^3+2pqx^2+(p^2+2q+c^2+1)x+2p=0.$$

It is clear that, in general, no linear relation connects the coefficients of this equation, so that $F_0(x)=1$; and the formula reduces to

$$\Sigma\int X dx = \Theta\left[\frac{1}{f_2(x)}\right]\Sigma\frac{f_1(x, y)}{\chi'(y)}\log\theta y$$

$$= \Theta\left[\frac{1}{1+nx^2}\right]\Sigma\frac{a+bx^2}{y}\log\{y-(1+px+qx^2)\}$$

$$= \frac{1}{n}\Theta\left[\frac{1}{\left(x+\frac{\iota}{\sqrt{n}}\right)\left(x-\frac{\iota}{\sqrt{n}}\right)}\right]\Sigma\frac{a+bx^2}{y}\log\{y-(1+px+qx^2)\}$$

where, as usual, $\iota=\sqrt{-1}$.

Therefore

$$n\Sigma\int X dx = -\frac{\sqrt{n}}{2\iota}\Sigma\frac{a-\frac{b}{n}}{y}\log\left\{y-\left(1-\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}$$

$$+\frac{\sqrt{n}}{2\iota}\Sigma\frac{a-\frac{b}{n}}{y}\log\left\{y-\left(1+\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}$$

$$-nC_{\frac{1}{x}}\frac{a+bx^2}{1+nx^2}\Sigma\frac{\log\{y-(1+px+qx^2)\}}{y}$$

Now the last term in general vanishes.

For

$$\Sigma\frac{\log\{y-(1+px+qx^2)\}}{y}=\Sigma\frac{\log(-qx^2)}{y}+\Sigma\frac{\log\left(1+\frac{px+1-y}{qx^2}\right)}{y}$$

and

$$\Sigma\frac{1}{y}=0$$

while $C_{\frac{1}{x}}$ in the term

$$\Sigma\frac{\log\left(1+\frac{px+1-y}{qx^2}\right)}{y} \quad \text{is} \quad \Sigma\frac{p}{qy}$$

and this vanishes.

Therefore the first term of the descending expansion of Σ involves x^{-2}, while that of $\frac{a+bx^2}{1+nx^2}$ begins with x^0;

wherefore

$$C_{\frac{1}{x}}\frac{a+bx^2}{1+nx^2}\Sigma\frac{\log\{y-(1+px+qx^2\}}{y}=0.^*$$

* There is an exceptional case if $\left.\begin{matrix}n=0\\ b\neq 0\end{matrix}\right\}$, for then the expansion of $\frac{a+bx^2}{1+nx^2}$ begins with x^2; and the $C_{\frac{1}{x}}$ is not necessarily zero.

Next, the two values of y under the sign Σ being

$$y=\pm\sqrt{\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{c^2}{n}\right)\right\}}, \text{ or say } \pm\frac{k}{n},$$

the first two terms in the expression of $n\Sigma\int Xdx$ compound into

$$-\frac{\sqrt{n}}{2\iota}\frac{na-b}{k}\log\frac{\left\{\frac{k}{n}-\left(1-\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}\left\{-\frac{k}{n}-\left(1+\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}}{\left\{-\frac{k}{n}-\left(1-\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}\left\{\frac{k}{n}-\left(1+\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}}$$

$$=\frac{\iota\sqrt{n}}{2}\frac{na-b}{k}\log\frac{(n-q)^2-(k+p\sqrt{n}\iota)^2}{(n-q)^2-(k-p\sqrt{n}\iota)^2}$$

which is easily rationalised, and gives

$$\sqrt{n}\frac{na-b}{k}\tan^{-1}\frac{2pk\sqrt{n}}{(n-q)^2+p^2n-k^2}$$

which, substituting for k^2 its value $(1+n)(c^2+n)$

$$=\sqrt{n}\frac{na-b}{k}\tan^{-1}\frac{\frac{2p}{q^2-c^2}k\sqrt{n}}{1+n\left\{1+\frac{p^2-(q+1)^2}{q^2-c^2}\right\}}$$

Now, if x_1, x_2, x_3 be the roots of the equation $E=0$, we get at once the relations

$$x_1x_2x_3=-\frac{2p}{q^2-c^2},$$

$$x_1^2+x_2^2+x_3^2-2-c^2x_1^2x_2^2x_3^2=2\frac{p^2-(q+1)^2}{q^2-c^2},$$

$$(1-x_1^2)(1-x_2^2)(1-x_3^2)=\left\{\frac{p^2-(q+1)^2}{q^2-c^2}\right\}^2.$$

We have then finally the following theorem.

If $$\psi(x)=\int_0 \frac{a+bx^2}{(1+nx^2)\sqrt{\{(1-x^2)(1-c^2x^2)\}}}dx$$

then, provided that x_1, x_2, x_3 are connected by the single relation

$$(2-x_1^2-x_2^2-x_3^2+c^2x_1^2x_2^2x_3^2)^2=4(1-x_1^2)(1-x_2^2)(1-x_3^2),$$

we have

$$\psi(x_1)+\psi(x_2)+\psi(x_3)=\sqrt{\frac{n}{(n+1)(n+c^2)}}\left(a-\frac{b}{n}\right)\tan^{-1}\frac{-\sqrt{\{n(n+1)(n+c^2)\}}x_1x_2x_3}{1+n\{1\pm\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)\}}}.^{*}$$

If we write $\sin\theta$ for x we have the corresponding expression

$$\sqrt{\frac{n}{(n+1)(n+c^2}}\left(a-\frac{b}{n}\right)\tan^{-1}\frac{-\sqrt{\{n(n+1)(n+c^2)\}}\sin\theta_1\sin\theta_2\sin\theta_3}{1+n(1\pm\cos\theta_1\cos\theta_2\cos\theta_3)}$$

for the sum of three integrals of the form

$$\int_0 \frac{a+b\sin^2\theta}{(1+n\sin^2\theta)\sqrt{(1-c^2\sin^2\theta)}}d\theta$$

whose variables are connected by the relation

$$(1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta_3-c^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3)^2=4\cos^2\theta_1\cos^2\theta_2\cos^2\theta_3\dagger.$$

From the formula just proved we can deduce without difficulty the well-known theorems connecting the elliptic functions of each order whose variables are connected by the equation

$$1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta_3-c^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3+2\cos\theta_1\cos\theta_2\cos\theta_3=0\ddagger$$

which is only another form of the familiar relation

$$\cos\theta_1=\cos\theta_2\cos\theta_3\pm\sin\theta_2\sin\theta_3\,\Delta\,\theta_1.$$

* It is here assumed that $n(n+1)(n+c^2)$ is positive. If this is not the case the imaginary $\tan^{-1}$ is replaced by a real logarithm.

† The exceptional case $\left.\begin{matrix}n=0\\ b\neq 0\end{matrix}\right\}$ in which there will be an additional term due to $C\frac{1}{x}$ must not be forgotten.

‡ We take the negative sign in the ambiguity.

For the first kind.

Here we put

$$a=1,\ b=0,\ n=0.$$

This does not fall under the exceptional case; and our formula gives

$$\mathrm{F}(\theta_1)+\mathrm{F}(\theta_2)+\mathrm{F}(\theta_3)=0.$$

For the second kind.

Here we put

$$a=1,\ b=-c^2,\ n=0.$$

This gives rise to the exceptional case.

The right-hand side of the formula vanishes. It remains to find the value of

$$-\mathrm{C}_{\frac{1}{x}}\Sigma\frac{1-c^2x^2}{y}\log\theta y$$

$$=-\mathrm{C}_{\frac{1}{x}}\frac{1-c^2x^2}{y_1}\log\frac{1+px+qx^2-y_1}{1+px+qx^2-y_1}$$

where $y_1=\sqrt{(1-x^2)(1-c^2x^2)}$

$$=2\mathrm{C}_{\frac{1}{x}}(1-c^2x^2\left\{\frac{1}{1+px+qx^2}+\frac{1}{3}\frac{y_1^{\,2}}{(1+px+qx^2)^3}+\ldots\right\}$$

which, clearly,

$$=-2c^2\mathrm{C}_{\frac{1}{x}}x^2\left\{\frac{1-\frac{p}{qx}+\ldots}{qx^2}+\frac{1}{3}\frac{(c^2x^4-\ldots)\left(1-\frac{3p}{qx}\ldots\right)}{q^3x^6}+\ldots\right\}$$

$$=2c^2\left(\frac{p}{q^2}+\frac{pc^2}{q^4}+\frac{pc^4}{q^6}+\ldots\right)$$

$$=\frac{2pc^2}{q^2-c^2}$$

$$=-c^2\sin\theta_1\sin\theta_2\sin\theta_3.$$

Therefore
$$\mathrm{E}(\theta_1)+\mathrm{E}(\theta_2)+\mathrm{E}(\theta_3)=-c^2\sin\theta_1\sin\theta_2\sin\theta_3.$$

For the third kind.

We have to write $a=1$, $b=0$, and get

$$\Pi(n,\theta_1)+\Pi(n,\theta_2)+\Pi(n,\theta_3)$$

$$=-\sqrt{\frac{n}{(n+1)(n+c^2)}}\tan^{-1}\frac{\sqrt{n(n+1)(n+c^2)}\sin\theta_1\sin\theta_2\sin\theta_3}{1+n(1-\cos\theta_1\cos\theta_2\cos\theta_3)}$$

(or the corresponding logarithmic expression if $n(n+1)(n+c^2)$ is negative).*

* CAYLEY, 'Elliptic Functions,' art. 132.

III. "ABEL'S *Theorem.*"

As a third example, consider a problem analogous to that of BOOLE, art. 20; but more easily reduced by ABEL'S theorem than by his.

Let

$$X \equiv \frac{\phi(x)}{\{\psi(x)\}^{\frac{m}{n}}}$$

where $\phi(x)$ is a rational integral or fractional function, $\psi(x)$ is a rational integral function, while m and n are positive integers prime to one another.

To this form any expression containing only a single term can be reduced.

Let

$$\chi \equiv y^n - \psi^m$$

while

$$\theta = \lambda_2 y - \lambda_1$$

λ_1 and λ_2 being rational integral functions: also let

$$\phi(x) \equiv \frac{\phi_1(x)}{\phi_2(x)}.$$

Then, eliminating,

$$E \equiv \lambda_1{}^n - \lambda_2{}^n \psi^m,$$

and, in general,

$$F_0(x) = 1.$$

So

$$X = \frac{\phi_1(x)}{\phi_3(x) y} = \frac{n y^{n-2} \phi_1(x)}{\phi_2(x) \chi'(y)}$$

so that

$$f_1(x, y) = n y^{n-2} \phi_1(x),$$

$$f_2(x) \quad = \phi_2(x).$$

Therefore

$$\Sigma \int X dx = \Theta \left[\frac{1}{\phi_2(x)}\right] \Sigma \frac{\phi_1(x)}{y} \log (\lambda_2 y - \lambda_1) + C$$

$$= \Theta[\phi(x)] \Sigma \frac{1}{y} \log (\lambda_2 y - \lambda_1) + C$$

But, if $1, \omega_1, \omega_2, \ldots \omega_{n-1}$ are the n^{th} roots of unity the values of y are

$$\psi^{\frac{m}{n}}, \psi^{\frac{m}{n}} \omega_1, \ldots \psi^{\frac{m}{n}} \omega_{n-1}.$$

So the last expression becomes (putting ω_0 for 1)

$$\Theta[\phi(x)] \psi^{-\frac{m}{n}} \left\{ \sum_0^{n-1} \omega \log \lambda_2 + \sum_0^{n-1} \omega \log \left(\omega \psi^{\frac{m}{n}} - \frac{\lambda_1}{\lambda_2} \right) \right\} + C$$

$$= \Theta[\phi(x)] \psi^{-\frac{m}{n}} \left\{ \sum_0^{n-1} \omega \log \left(\omega \psi^{\frac{m}{n}} - \frac{\lambda_1}{\lambda_2} \right) \right\} + C$$

since

$$\sum_0^{n-1} \omega = 0.$$

As a particular case of this result what is often called ABEL'S Theorem may be adduced.

Let

$$X=\frac{f(x)}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}.$$

We have to write in the previous work

for	$\phi(x)$	$\frac{f(x)}{x-a}$
for	$\psi(x)$	$\phi_1(x)\phi_2(x)$
for	$\frac{m}{n}$	$\frac{1}{2}$

The right-hand side becomes

$$\Theta\left[\frac{f(x)}{x-a}\right]\frac{1}{\sqrt{\phi_1(x)\phi_2(x)}}\sum_0^1\omega\log\left(\omega\sqrt{\phi_1(x)\phi_2(x)}-\frac{\lambda_1}{\lambda_2}\right)+C.$$

The two values of ω are $+1$, -1.

Therefore the above

$$=\Theta\left[\frac{f(x)}{x-a}\right]\frac{1}{\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{\lambda_1-\lambda_2\sqrt{\phi_1(x)\phi_2(x)}}{\lambda_1+\lambda_2\sqrt{\phi_1(x)\phi_2(x)}}.$$

This assumes a more symmetrical shape if, with ABEL, we write, not $\sqrt{\phi_1(x)\phi_2(x)}=\frac{\lambda_1}{\lambda_2}$, but $\sqrt{\frac{\phi_2(x)}{\phi_1(x)}}=\frac{\lambda_1}{\lambda_2}$; so that $\sqrt{\phi_1(x)\phi_2(x)}=\frac{\lambda_1\phi_1(x)}{\lambda_2}$.

With this alteration we get

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}=\Theta\left[\frac{f(x)}{x-a}\right]\frac{1}{\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{\lambda_1\sqrt{\phi_1(x)}-\lambda_2\sqrt{\phi_2(x)}}{\lambda_1\sqrt{\phi_1(x)}+\lambda_2\sqrt{\phi_2(x)}}+C$$

$$=\frac{f(a)}{\sqrt{\phi_1(a)\phi_2(a)}}\log\frac{\lambda_1(a)\sqrt{\phi_1(a)}-\lambda_2(a)\sqrt{\phi_2(a)}}{\lambda_1(a)\sqrt{\phi_1(a)}+\lambda_2(a)\sqrt{\phi_2(a)}}$$

$$-C_{\frac{1}{x}}\frac{f(x)}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{\lambda_1(x)\sqrt{\phi_1(x)}-\lambda_2(x)\sqrt{\phi_2(x)}}{\lambda_1(x)\sqrt{\phi_1(x)}+\lambda_2(x)\sqrt{\phi_2(x)}}+C$$

which is the well-known theorem referred to.

We see it to be only a particular case of a particular case of the theorem called in this paper ABEL'S Theorem.

SECTION II.

12. The expression (in a form algebraic or logarithmic) of the sum $\Sigma\int X dx$ having been shown to exist, and having in fact been found, ABEL proceeds, in his art. 5, to investigate the condition that this expression should become a constant. Of the possibility of this we have been assured by the result of the first example and of the first case of the second example of art. 11. This investigation, as subordinate to the main purpose, may be conveniently postponed to the second principal inquiry with which the memoir is concerned.

This inquiry presents itself in two forms.

I. Mention was made at the outset of the "requisite algebraical laws" which connect the variables when the summation desired can be effected. And in the case of the elliptic functions we have found that in order to express the sum of three functions it is requisite that the variables should be connected by a *single* relation. We are naturally led to investigate the number of relations necessary for the same effect in the case of more complicated forms. This number, it must be said, depends not at all on the *number* of the functions we consider but only on their *form*.

II. We may also consider the matter thus :—

Representing by $\psi(x)$ the integral $\int X dx$, we have shown how to express, by the use of an operative symbol Θ, the sum

$$\psi(x_1)+\psi(x_2)+\ldots+\psi(x_\mu)$$

where $x_1, x_2, \ldots x_\mu$ are the roots of an equation

$$F(x)=0.$$

Now this equation involves a number, α, of arbitrary quantities $a_1, a_2, a_3 \ldots$

Its μ roots are functions of these α quantities. We can then find expressions for $a_1, a_2, \ldots$, in terms of α of these roots, say $x_1, x_2, \ldots x_\alpha$; and substituting these expressions in those which give $x_{\alpha+1} \ldots x_\mu$ shall have these $\mu-\alpha$ roots determined as functions of the other α.*

The result then is an expression for the sum of a series of functions

$$\psi(x_1)+\ldots+\psi(x_\alpha),$$

* This is most conveniently effected by

(1) solving for $a_1, a_2, \ldots$ the α equations—linear in a's—

$$\theta(y_1)=0, \qquad \theta(y_2)=0 \ldots, \qquad \theta(y_\alpha)=0,$$

where the equation $\theta(y_1)$ is the factor of E which supplies the factor $x-x_1$ to $F(x)$, and

(2) substituting the values so obtained in $F(x)$, which then becomes divisible by

$$(x-x_1)(x-x_2)\ldots(x-x_\alpha),$$

and gives as quotient an equation of the degree $\mu-\alpha$ whose coefficients are rational integral functions of (x_1, y_1), &c., and whose roots are the quantities $x_{\alpha+1}, x_{\alpha+2}, \ldots x_\mu$ which it is required to determine.

$x_1 \ldots x_\alpha$ being any quantities whatever, in terms of an algebraic function and a number of functions of the same form whose variables are themselves definite functions of the quantities $x_1, x_2, \ldots x_\alpha$.

The question then arises, What is the smallest number of functions in terms of which the sum may be expressed? and can the sum of *any* number of functions be expressed in terms of this smallest number?*

13. *Required the least value of which the difference between the number of roots possessed by the 'equation of the limits' and the number of constants introduced by the 'equation of condition' is susceptible.*

This difference is expressed by $\mu-\alpha$. We must put each term under a different form.

(i.) For α.

Let us express the index of the highest power of x in a function $J(x)$, supposed rational and integral, by the symbol $\overline{J(x)}$.

Then in general the number of coefficients in $J(x)$ is $\overline{J(x)}+1$: and as in θ one coefficient may without loss of generality be written unity

$$\alpha=\text{number of coefficients in } \theta(\equiv q_{n-1}y^{n-1}+\ldots+q_0)$$
$$=\Sigma\bar{q}+n-1.$$

Two corrections must be introduced.

For the existence of each linear factor of F_0 implies a linear relation between the α's, and diminishes the independent number by unity. We have on this account to subtract $\overline{F_0}$. It may happen, however, that the particular form of the function renders the number of necessary relations less. Write then $\overline{F_0}-A$ as the quantity to be subtracted.

Suppose again that some of the constants are so chosen as to reduce the degree of E.†

In general μ and α are thus equally reduced; but it may happen that the form of the function renders necessary a less number of conditions. If this lessens μ by a number greater by B than the lessening of α we have to use instead of $\overline{F_0}-A$, $\overline{F_0}-A-B$.

We will however for the present drop the A and B, which would appear without alteration throughout the process, and replace them at its conclusion in the shape of a correction to the result.

* In an earlier memoir (ABEL's works, vol. ii., xi.), this question is dismissed with the remark "il n'est pas difficile de se convaincre que, quelque soit le nombre μ on peut toujours faire en sorte que $n-\mu$ devienne independant de μ." Here the actual value of this constant is investigated.

† For example, in the case of p. 725, we put $\sqrt{1-x^2.1-c^2x^2}=1+px+qx^2$, and the assumption of unity as the first term on the right reduced the resulting equation from a quartic to a cubic.

We have then

$$\alpha = \Sigma\overline{q} + n - 1 - \overline{F_0}.$$

(ii.) *For* μ.

Since

$$\theta(y_1)\theta(y_2) \ldots \theta(y_n) = F_0 F$$

it follows that

$$\Sigma\overline{\theta(y)} = \overline{F_0} + \overline{F} = \overline{F_0} + \mu$$

So

$$\mu - \alpha = \Sigma\overline{\theta(y)} - \Sigma\overline{q} - n + 1.*$$

Now

$$\begin{aligned}\overline{\theta(y)} &\geqq \overline{q_r y^r}\\ &\geqq \overline{q_r} + r\overline{y}\end{aligned}$$

and it becomes necessary to find $\overline{y_1}, \overline{y_2}, \ldots \overline{y_n}$.

14. We require the following Lemma.

The quantities $\overline{y_1}, \overline{y_2}, \ldots \overline{y_n}$, *are in general equal in sets.*

For let $\overline{y_1} = \frac{m_1}{\mu_1}$; this being a fraction in its lowest terms (and we will take the denominator positive).

Then one root of χ being, when expanded in descending powers of x,

$$y = Ax^{\frac{m_1}{\mu_1}} + \ldots$$

the expressions

$$\begin{aligned}y &= A\omega_1 x^{\frac{m_1}{\mu_1}} + \ldots\\ y &= A\omega_2 x^{\frac{m_1}{\mu_1}} + \ldots\\ y &= \&c.\end{aligned}$$

(where $1, \omega_1, \omega_2 \ldots$ are the μ^{th} roots of unity) are also roots, and if these are $y_2, y_3 \ldots$ we have $\overline{y_1} = \overline{y_2} = \ldots$, the number equated being clearly a multiple of μ_1. Let it be $n_1\mu_1$; and write

$$\begin{aligned}&\overline{y_1} = \overline{y_2} = \ldots = \overline{y_{k_1}} && \text{where } k_1 = n_1\mu_1\\ &\overline{y_{k_1+1}} = \ldots = \overline{y_{k_2}} && \text{where } k_2 - k_1 = n_2\mu_2\\ &\qquad \&c. = \&c.\\ &\overline{y_{k_{l-1}+1}} = \ldots = \overline{y_{k_l}} && \text{where } k_l - k_{l-1} = n_l\mu_l\\ & && \text{and } k_l = n\dagger\end{aligned}$$

* Here $\overline{\theta(y)}$ means the degree of $\theta(y)$ when rendered a function of x by substitution for y from the equation $\chi(y) = 0$.

† This lemma is the second of the theorems laid down by ABEL in his important memoir "Sur la résolution algébrique des équations," of which consists the last article (it was never finished) in the second volume of his works.

Also let us write, for shortness,

$$\frac{m_1}{\mu_1}=\sigma_1,\ \frac{m_2}{\mu_2}=\sigma_2,\ \ldots\ \frac{m_l}{\mu_l}=\sigma_l;$$

and let these be in descending order of magnitude, so that

$$\sigma_1>\sigma_2>\sigma_3\ldots>\sigma_l.$$

We have then n_1 sub-sets, each of μ_1 terms, with index $\frac{m_1}{\mu_1}$, n_2 sub-sets each of μ_2 terms, with index $\frac{m_2}{\mu_2}$, and so on.

These quantities m_1, μ_1; m_2, μ_2; &c., can be speedily determined when χ is given by NEWTON'S method.

Thus, write $\mathrm{A}x^\sigma$ for y in the equation, and determine σ by the condition that in the resulting function of x the indices in two or more terms may be equal and greater than in any other term (while the condition that the sum of these terms shall vanish will determine A).*

These conditions are obviously necessary for the existence of a root $y=\mathrm{A}x^\sigma+\ldots$: and it is easy to prove directly that we can thus determine values of the quantities σ unique, and in descending order.

For suppose the indices after substitution to be $n\sigma$; $(n-1)\sigma+a_1$; $(n-2)\sigma+a_2$; ...

Then putting

$$n\sigma=(n-k)\sigma+a_k$$

we have

$$\sigma=\frac{a_k}{k};$$

* As an example, suppose that y is determined by the cubic

$$\chi\equiv y^3+p_2y^2+p_1y+p_0=0$$

while

$$\overline{p_2}=1;\ \overline{p_1}=3;\ \overline{p_0}=2.$$

Writing $\mathrm{A}x^\sigma$ for y the exponents are

$$3\sigma,\ 2\sigma+1,\ \sigma+3,\ 2.$$

It is clear that the conditions are satisfied by making $3\sigma=\sigma+3$, *i.e.*, $\sigma=\frac{3}{2}$, while a quadratic is obtained for A, so that there are two corresponding terms and $\overline{y_1}=\overline{y_2}$.

They are also satisfied by making $\sigma+3=2$, *i.e.*, $\sigma=-1$, and a simple equation is obtained for A.

We have, then,

$$\frac{m_1}{\mu_1}\equiv\sigma_1=\tfrac{3}{2};\quad n_1=1,$$

$$\frac{m_2}{\mu_2}\equiv\sigma_2=-1;\ n_2=1.$$

and if we choose k so that $\frac{a_k}{k}$ is the greatest of the series $\frac{a_1}{1}, \frac{a_2}{2} \ldots$, we have, determined as a unique value, what we will provisionally call σ_1.

Next put

$$(n-k)\sigma+a_k=(n-s)\sigma+a_s$$

whence

$$\sigma=\frac{a_s-a_k}{s-k}$$

Now this value is to make

$$(n-k)\sigma+a_k>(n-t)\sigma+a_t$$

or

$$(t-k)a_s-(s-k)a_t>(t-s)a_k,$$

and since by interchanging s and t we get the contradictory of this inequality, it is impossible that by putting

$$(n-k)\sigma+a_k=(n-t)\sigma+a_t$$

each of these could be made $>(n-s)\sigma+a_s$.

Therefore the second step is also unique; and

$$\frac{a_s-a_k}{s-k}<\frac{a_k}{k} \quad \text{since } \frac{a_s}{s}<\frac{a_k}{k},$$

so that the second σ is less than the first and may be called σ_2.

Now, resuming the process of art. 13, divide the terms of the expression

$$\theta(y)\equiv q_{n-1}y^{n-1}+q_{n-2}y^{n-2}+\ldots+q_1y+q_0$$

into sets: calling the first k_1 of them the *first set*, the next k_2-k_1 the *second set*, and so on, the last k_l-k_{l-1} constituting the l^{th} set.

Also call that term of the first set in which when y_1 is written for y the highest resulting index of x is the largest the *major term of the first set*, call that term of the second set in which on the substitution of y_2 the same happens the *major term of the second set*, and so on.

Then I proceed

(i) to show that by a proper choice of the quantities $\overline{q_{n-1}}, \overline{q_{n-2}}, \ldots \overline{q_1}, \overline{q_0}$, which are at our disposal, we can make the major term of the first set an *absolute* major (for the substitution y_1), *i.e.*, furnish a higher index of x than is furnished by *any* other term; the major term of the second set an absolute major (for the substitution y_2), and so on,

(ii) to show that the condition of (i) is necessary in order that $\mu-\alpha$ may have the smallest value of which it is susceptible.

(iii) to find this value.

The proof of (i) is most simply conducted by successively investigating the conditions

(a) that the major term of any (say the r^{th}) set should furnish a higher power of x (for the substitution y_r) than any other *major term* furnishes,

(b) that this major term should furnish a higher power than any other not-major term furnishes.

In investigating (b) the conditions of (a) are to be supposed to hold. It will only be found necessary to supply to them a slight additional restriction in order to satisfy (b).

17. The condition for (a) is that whatever values (of course lying between 0 and $n-1$ inclusive) are given to r and s we should have

$$\overline{q_{\rho_r}}+\rho_r\sigma_r>\overline{q_{\rho_s}}+\rho_s\sigma_r,$$

where we have taken $q_{\rho_r}\, y^{\rho_r}$ to be the major term of the r^{th} set.

We will write this, for brevity, in the form

$$[\rho_r]+\rho_r\sigma_r>[\rho_s]+\rho_s\sigma_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (A)$$

so that

$$[\rho_r]\equiv\overline{q}_{\rho_r}$$

If we make successively the substitutions

$$\left.\begin{array}{l} r=m+1 \\ s=m \end{array}\right\} \qquad \left.\begin{array}{l} r=m \\ s=m+1 \end{array}\right\}$$

we find that the above inequality requires the following :—

$$[\rho_{m+1}]-[\rho_m]\begin{array}{l}>(\rho_m-\rho_{m+1})\sigma_{m+1} \\ <(\rho_m-\rho_{m+1})\sigma_m\end{array}$$

If then we write

$$[\rho_{m+1}]-[\rho_m]=(\rho_m-\rho_{m+1})\tau_m$$

we have

$$\tau_m\begin{array}{l}>\sigma_{m+1} \\ <\sigma_m\end{array}$$

If we use also for $\rho_m-\rho_{m+1}$ the abbreviation $\delta\rho_m$ we have

$$[\rho_{m+1}]-[\rho_m]=\delta\rho_m.\tau_m$$

and it follows that

$$\begin{aligned}[\rho_m]&=[\rho_1]+\delta\rho_1.\tau_1+\ldots+\delta\rho_{m-1}.\tau_{m-1} \\ &=[\rho_1]+\sum_{k=1}^{k=m-1}\delta\rho_k.\tau_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (B)\end{aligned}$$

The condition expressed by this equation is then necessary if the inequality (A) is to hold.

It is also sufficient, as is easily shown.

For from it we obtain

if $r>s$

$$\begin{aligned}[\rho_r]-[\rho_s]&=\delta\rho_s.\tau_s+\delta\rho_{s+1}.\tau_{s+1}+\ldots+\delta\rho_{r-1}.\tau_{r-1}\\ &>(\delta\rho_s+\delta\rho_{s+1}+\ldots+\delta\rho_{r-1})\tau_{r-1}\\ &>(\rho_s-\rho_r)\tau_{r-1}\\ &>(\rho_s-\rho_r)\sigma_r\end{aligned}$$

if $r<s$

$$\begin{aligned}[\rho_r]-[\rho_s]&=-\delta\rho_r.\tau_r-\delta\rho_{r+1}.\tau_{r+1}-\ldots-\delta\rho_{s-1}.\tau_{s-1}\\ &>-(\delta\rho_r+\delta\rho_{r+1}+\ldots+\delta\rho_{s-1})\tau_r\\ &>(\rho_s-\rho_r)\sigma_r.\end{aligned}$$

The relation (B) (in which $[\rho_1]$ is entirely arbitrary and the τ's are only subject to the necessity of lying between consecutive σ's) expresses the necessary and sufficient condition for the satisfaction of (a).

18. Let us next examine (b).

The condition is expressed by the inequality

$$[\rho_m]+\rho_m\sigma_m>[\alpha]+\alpha\sigma_m$$

where α is any term of the series 0, 1, ... $(n-1)$ which is not one of the ρ's.

Let α belong to the λ^{th} set so that

$$[\alpha]+\alpha\sigma_\lambda<[\rho_\lambda]+\rho_\lambda\sigma_\lambda$$

and let

$$[\alpha]+\alpha\sigma_\lambda=[\rho_\lambda]+\rho_\lambda\sigma_\lambda-A_\alpha \quad \ldots\ldots\ldots \quad \text{(C)}$$

A_α being a positive quantity.

We have then to make

$$[\rho_m]+\rho_m\sigma_m>[\rho_\lambda]+\rho_\lambda\sigma_\lambda+\alpha(\sigma_m-\sigma_\lambda)-A_\alpha \quad \ldots\ldots \quad \text{(D)}$$

Now this inequality clearly holds when $m=\lambda$. Again it holds when $m=\lambda+1$ provided that

$$[\rho_{\lambda+1}]-[\rho_\lambda]>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\rho_\lambda-\rho_{\lambda+1})\tau_\lambda>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha.$$

But this will always be possible if

$$(\rho_\lambda-\rho_{\lambda+1})\sigma_\lambda>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\alpha-\rho_{\lambda+1})(\sigma_\lambda-\sigma_{\lambda+1})>-A_\alpha,$$

a relation which is always true since $\alpha-\rho_{\lambda+1}$ and $\sigma_\lambda-\sigma_{\lambda+1}$ are both positive.

Once more, it holds when $m=\lambda-1$ if

$$[\rho_{\lambda-1}]-[\rho_\lambda]>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda-1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\rho_\lambda-\rho_{\lambda-1})\tau_{\lambda-1}>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda-1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\rho_{\lambda-1}-\rho_\lambda)\tau_{\lambda-1}<\rho_{\lambda-1}\sigma_{\lambda-1}-\rho_\lambda\sigma_\lambda-\alpha(\sigma_{\lambda-1}-\sigma_\lambda)+A_\alpha$$

and this can, as in the last case, be shown to be always possible.

Now if the inequality (D) holds, m being greater than λ, it will hold when for m we write $m+1$ provided that

$$[\rho_{m+1}]-[\rho_m]+\rho_{m+1}\sigma_{m+1}-\rho_m\sigma_m>\alpha(\sigma_{m+1}-\sigma_m)$$

i.e., if

$$\rho_m(\sigma_{m+1}-\sigma_m)>\alpha(\sigma_{m+1}-\sigma_m)$$

but

$$\rho_m<\alpha, \quad \sigma_m>\sigma_{m+1}$$

therefore this relation does hold.

But the inequality (D) is true when $m=\lambda+1$. It is therefore true for all larger values of m.*

It can similarly be shown that if the inequality holds, m being less than λ, it will hold when for m we write $m-1$; and that, since it holds when $m=\lambda-1$, it holds for all less values of m.

It is therefore proved universally.

We observe that, as was stated at the outset, the consideration of the case (b) has only introduced a restriction into the conditions of the case (a)—viz.: that the τ's are no longer merely subject to the necessity of lying between consecutive σ's, but must satisfy the closer conditions expressed by the inequalities

$$(\rho_\lambda-\rho_{\lambda+1})\tau_\lambda \begin{array}{l} >\rho_\lambda\sigma_\lambda-\rho_{\lambda+1}\sigma_{\lambda+1}+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha \\ <\rho_\lambda\sigma_\lambda-\rho_{\lambda+1}\sigma_{\lambda+1}+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)+A_\alpha \end{array}\Bigg\} \quad . \quad . \quad . \quad . \quad (E)$$

where in the first line α denotes any one of the numbers of the λ^{th}, in the second any one of the $(\lambda+1)^{th}$ set.

19. We have next to consider the second proposition of page 735, viz.: The condition of (i) is necessary if $\mu-\alpha$ is to have its smallest value.

* It must be observed that this method of proof could not be used to deduce the case $m+1$, $\lambda+1$ from the case m, λ; for it would not be necessarily true that ρ_m is less than α.

Writing down a series of equations similar to (C) we have

$$\left.\begin{array}{l}\left.\begin{array}{lll}\overline{\theta(y_1)} & =[\rho_1]+\rho_1\sigma_1=[n-1] & +(n-1)\sigma_1 \quad +A_{n-1} \\ \overline{\theta(y_2)} & =[\rho_1]+\rho_1\sigma_1=[n-2] & +(n-2)\sigma_1 \quad +A_{n-2} \\ \&c. & =\&c. \qquad =\&c. & \\ \overline{\theta(y_{k_1})} & =[\rho_1]+\rho_1\sigma_1=[n-k_1] & +(n-k_1)\sigma_1 \quad +A_{n-k_1}\end{array}\right] \text{First set} \\ \left.\begin{array}{l}\overline{\theta(y_{k_1+1})}=[\rho_2]+\rho_2\sigma_2=[n-k_1-1]+(n-k_1-1)\sigma_2+A_{n-k_1-1} \\ \&c. \quad =\&c. \qquad =\&c.\end{array}\right] \text{Second set} \\ \&c. \quad =\&c. \qquad =\&c. \qquad]\ \&c.\end{array}\right\} \quad . \quad (F)$$

and, adding all these lines together,

$$\Sigma\overline{\theta(y)}=[n-1]+[n-2]+\ldots+[0]+(n-1+\ldots+n-k_1)\sigma_1$$
$$+(n-k_1-1+\ldots+n-k_2)\sigma_2+\ldots+\Sigma A$$

or

$$\Sigma\overline{\theta y}-\Sigma\overline{q}=(n-1+\ldots+n-k_1)\sigma_1+(n-k_1-1+\ldots+n-k_2)\sigma_2+\ldots+\Sigma A$$

Now, if the condition of (i) were not satisfied, some at least of the signs of equality connecting the first and second vertical columns must have been replaced by the sign $>$; and as those between the second and third column would have remained as before, the equality at the head of this page would have become an inequality—*i.e.*, the value of $\Sigma\overline{\theta y}-\Sigma\overline{q}$ would have been greater than it is—*i.e.*, $\mu-\alpha$ would have been greater.

It only remains to consider the term ΣA.

The smaller we can make this sum, and therefore, all the terms being positive, the smaller we can make each term, the less will be our value of $\mu-\alpha$.

Now from the general equation

$$[\rho_\lambda]+\rho_\lambda\sigma_\lambda=[\alpha]+\alpha\sigma_\lambda+A_\alpha$$

we see that, since $[\rho_\lambda]$ and $[\alpha]$ are integers, A_α consists in general of two parts—an integer and the proper fraction which added to $(\alpha-\rho_\lambda)\sigma_\lambda$ will make it integral.

Now we can make the integral part vanish for every value of α; for to do so will only require a relation between the major term and the other terms of each set; so that, given the degree in x of the major term, those of the others in its set can be written down.

As the conditions (i) only connect with one another the major terms of different sets, this last condition is independent of them and can always be satisfied.

20. To find the value of $\mu-\alpha$ we must investigate the fractional parts.

Considering any set (say the λ^{th}), they are, with the notation of the Lemma (p. 746), of the form

$$\epsilon^{\frac{(\rho_\lambda-\alpha)m_\lambda}{\mu_\lambda}}$$

where α takes each value from $n-k_{\lambda-1}-1$ to $n-k_\lambda$; and $k_\lambda-k_{\lambda-1}=n_\lambda\mu_\lambda$.

But m_λ and μ_λ are prime to each other.

Therefore, by the result of the Lemma, the sum

$$=n_\lambda\frac{\mu_\lambda-1}{2}.$$

So then finally, giving to ΣA its least value, we have

$$\Sigma\overline{\theta y}-\Sigma\overline{q}=\{(n-1)+\ldots+(n-k_1)\}\sigma_1+\{(n-k_1-1)+\ldots+(n-k_2)\}\sigma_2+\ldots+\Sigma\tfrac{1}{2}n(\mu-1)$$

This expression

$$=\tfrac{1}{2}k_1\sigma_1(2n-k_1-1)+\tfrac{1}{2}k_2\sigma_2(2n-k_1-k_2-1)+\ldots+\Sigma\tfrac{1}{2}n(\mu-1).$$

Now $k_1=n_1\mu_1$; $k_2=n_1\mu_1+n_2\mu_2$; &c. = &c.; $n=k_l=n_1\mu_1+n_2\mu_2+\ldots+n_l\mu_l$.

Substituting we obtain

$$n_1m_1\left(\frac{n_1\mu_1-1}{2}+n_2\mu_2+\ldots+n_l\mu_l\right)$$
$$+n_2m_2\left(\frac{n_2\mu_2-1}{2}+n_3\mu_3+\ldots+n_l\mu_l\right)$$
$$+\ldots$$
$$+\Sigma\tfrac{1}{2}n(\mu-1)$$
$$=\sum_{s>r}n_rm_rn_s\mu_s+\tfrac{1}{2}\Sigma n^2m\mu+\tfrac{1}{2}\Sigma n\mu-\tfrac{1}{2}\Sigma n-\tfrac{1}{2}\Sigma nm.$$

Now, returning to the values of art. 13 and inserting the numbers A and B for the correction there explained and writing instead of $\Sigma n\mu$ its equivalent n, we have the result following.

The least number of functions in terms of which the sum of any number may be expressed is independent of everything but the form of the function considered (i.e., the form of y given as a function of x by the equation $\chi(y)=0$), *and if this equation has*

$n_1\mu_1$ *roots of the form* $y = Cx^{\frac{m_1}{\mu_1}} + \ldots$, $n_2\mu_2$ *of the form* $y = Cx^{\frac{m_2}{\mu_2}} + \ldots$, *and so on, the number is*

$$= \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n - \tfrac{1}{2}n + 1 - A - B \quad . \quad . \quad . \quad . \quad (G)$$

(the last two terms $-A-B$ corresponding to a correction which is in general zero).*

21. It may be well to render these methods and formulæ plainer by applying them to an example. We will choose for this purpose the simple case already considered in the note on p. 734.

Our last formula for the value of $\mu-\alpha$ gives, if we assume that, as in general is the case, the values of A and B are zero, writing

$$\begin{array}{lll} m_1=3 & & \\ & n_1=1 & \sigma_1=\tfrac{3}{2} \\ \mu_1=2 & & \end{array}$$

$$\begin{array}{lll} m_2=-1 & & \\ & n_2=1 & \sigma_2=-1 \\ \mu_2=1 & & \end{array}$$

$$\begin{aligned} \mu-\alpha &= 3(1)+\tfrac{1}{2}(6-1)-\tfrac{1}{2}(3-1)-\tfrac{1}{2}(2)-\tfrac{3}{2}+1 \\ &= 3. \end{aligned}$$

We will next find the values of $\bar{q}_0$, $\bar{q}_1$, $\bar{q}_2$, or, as we have written them, [0], [1], [2]. We have

$$\rho_1 = 2 \text{ or } 1$$
$$\rho_2 = 0$$

Let us take $\rho_1 = 2$.†

Then, by the formulæ (F),

$$\begin{aligned} &[2]=[2] \\ &[1]=[2]+\tfrac{3}{2}-A_1; \text{ so } A_1=\tfrac{1}{2};\ [1]=[2]+1 \\ &[0]=[0] \end{aligned}$$

* In the most simple case, when

$$y^n + p_1 y^{n-1} + \ldots + p_{n-1}y + p_n$$

is the completely general function $(y, x, 1)^n$

$$n_1 = n, \quad m_1 = 1, \quad \mu_1 = 1$$

and

$$\begin{aligned} \mu-\alpha &= \tfrac{1}{2}n^2 - \tfrac{3}{2}n + 1 = \tfrac{1}{2}(n-1)(n-2) \\ &= \text{deficiency of general } n\text{-tic curve.} \end{aligned}$$

This is a case of the result shown by Professor CAYLEY in the Addition to be universally true.

† We might have taken $\rho_1 = 1$ with a similar result. This multiplicity of solution will generally occur.

and by the definition of τ

$$[0]-[2]=2\tau_1$$

while from the conditions (E)

$$2\tau_1>3-\tfrac{5}{2}\alpha-A_{\alpha}, \text{ i.e., } >3-5, \text{ or } 3-\tfrac{5}{2}-\tfrac{1}{2}$$
$$<3-\tfrac{5}{2}\alpha+A_{\alpha}, \text{ i.e., } <3$$

so that

$$2\tau_1>0<3$$

whence

$$[0]=[2],\ [2]+1,\ [2]+2,\ [2]+3,$$

and so if the degree of q_2 be denoted by θ that of q_1 is $\theta+1$; and that of q_0 may be either θ, $\theta+1$, $\theta+2$, or $\theta+3$.

We have, then, by art. 13 (i)

$$\alpha=[0]+[1]+[2]+2=3\theta+3,\ 3\theta+4,\ 3\theta+5, \text{ or } 3\theta+6$$

while

$$\mu=n_1\mu_1\{[\rho_1]+\rho_1\sigma_1\}+n_2\mu_2\{[\rho_2]+\rho_2\sigma_2\}$$
$$=2(\theta+3)+\{\theta,\ \theta+1,\ \theta+2,\ \theta+3\}$$
$$=3\theta+6,\ 3\theta+7,\ 3\theta+8,\ 3\theta+9$$

So that, as on the last page,

$$\mu-\alpha=3.$$

We have proved then that the sum of any number of integrals of the form indicated by the fact that they are rationalized by the introduction of y, where

$$y^3+p_2y^2+p_1y+p_0=0,$$

can be reduced to the sum of *three;* the equation of condition being $q_2y^2+q_1y+q_0=0$, where $\overline{q_1}=\overline{q_2}+1$, and $\overline{q_0}$ lies between $\overline{q_2}$ and $\overline{q_2}+3$ inclusive.

SECTION III.

22. We have shown that the sum of any number whatever of similar functions such as are discussed in this paper can be reduced to an expression algebraical or logarithmic added to a fixed number of such functions whose variables are functions of the variables of the given functions, this fixed number depending only on the form of function considered.

From this a more general theorem may be shown to follow, viz.: that *a similar* *expression may be found for the sum of any number of such functions each multiplied by any rational number positive or negative, integral or fractional.*

If all the rational numbers are positive and integral the theorem follows at once by supposing the functions whose sum we have shown how to express to be equal in sets. And this suggests the method of treating the general case when the numbers are any whatever.

Let $\theta \equiv \mu - \alpha =$ fixed number to which the sum of the functions has been shown to be reducible.

Then, by previous work (compare pp. 731, 732).

$$\psi_1(x_1)+\psi_2(x_2)+\ldots+\psi_\alpha(x_\alpha)=v-\{\psi_{\alpha+1}(x_{\alpha+1})+\ldots+(\psi_{\alpha+\theta})(x_{\alpha+\theta})\}$$
$$\psi_1(\text{X}_1)+\psi_2(\text{X}_2)+\ldots+\psi_{\alpha'}(\text{X}_{\alpha'})=\text{V}-\{\psi_{\alpha'+1}(\text{X}_{\alpha'+1})+\ldots+\psi_{\alpha'+\theta}(\text{X}_{\alpha'+\theta})\}$$

where α and α' are any numbers whatever; $x_{\alpha+1}\ldots x_{\alpha+\theta}$ are functions of $x_1\ldots x_\alpha$; and $\text{X}_{\alpha'+1}\ldots\text{X}_{\alpha'+\theta}$ of $\text{X}_1\ldots\text{X}_\alpha$, and v, V are algebraical and logarithmic functions.

Subtract: and let the last θ of the terms on the left-hand side of the second be (both as to functional form and variable) identical with those in the bracket in the first. Then, writing β for $\alpha'-\theta$, we have

$$\psi_1(x_1)+\ldots+\psi_\alpha(x_\alpha)-\psi_1(\text{X}_1)-\ldots-\psi_\beta(\text{X}_\beta)=v-\text{V}+\{\psi_{\alpha'+1}\text{X}_{\alpha+1}+\ldots+\psi_{\alpha+\theta}(\text{X}_{\alpha+\theta})\}.$$

Equate all the functions on the right to zero.

This will give θ relations between the x's and X's.

Now making the functions on the left equal in sets, and dividing by any requisite integer, we have a result which may be written

$$h_1\phi_1(y_1)+h_2\phi_2(y_2)+\ldots+h_m\phi_m(y_m)\equiv\text{W}$$

where the ϕ's are similar functions, m is any number whatever, W is an algebraical and logarithmic function of the y's, which are themselves connected by θ relations, and the h's are any numbers whatever.

If we express θ of these variables as functions of the rest and call them z's, putting n for $m-\theta$, we can write

$$h_1\phi_1(y_1)+h_2\phi_2(y_2)+\ldots+h_n\phi_n(y_n)=w+k_1\phi'_1(z_1)+\ldots+k_\theta\phi'_\theta(z_\theta).$$

Or making, as we may, the k's each $=$ unity we have shown how to find the expression required.*

* The subscript letters attached here, and not before, to the functional symbols introduce no novelty. They are only intended to suggest the fact that what we have written $\psi(x_1)$, $\psi(x_2)\ldots$ are really $\psi(x_1, y_1)$, $\psi(x_2, y_2),\ldots$; while y_1 and $y_2\ldots$ are not necessarily the same functions of x_1, $x_2\ldots$ This has not been hitherto overlooked, it is only more clearly put in evidence now.

23. We may conveniently investigate at this point, as a corollary to previous work, the conditions necessary that the 'algebraic and logarithmic function' often referred to already should become a constant; in other words, that the term involving Θ in the expression of ABEL's theorem should disappear, and with it the arbitrary quantities $a_1, a_2 \ldots$

We will assume $F_0(x)=1$ for the sake of simplicity, and have therefore the formula of Art. 9.

The first condition is that

$$f_2(x)=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

for otherwise the terms contributed by it to Θ will introduce the arbitrary quantities a.

Next, we must have

$$C_{\frac{1}{x}} \Sigma \frac{f_1(x,y)}{\chi'(y)} \log \theta y = 0$$

or, which comes to the same effect,

$$C_{\frac{1}{x}} \Sigma \frac{f_1(xy)}{\chi'(y)} \cdot \frac{\delta\theta y}{\theta y} = 0$$

and since $\overline{\delta\theta y}=\overline{\theta y}$, δ indicating differentiation with respect to a's, and consequently not altering the degree of a function in x,

$$\overline{\left(\frac{\delta\theta y}{\theta y}\right)}=0$$

and the condition to be satisfied is

$$\overline{\left(\frac{f_1(xy)}{\chi'(y)}\right)}<-1$$

when, for y, any whatever of the series $y_1, y_2 \ldots y_n$ has been substituted.

Now $f_1(x\,y)$, being integral and rational, can be expanded in the form

$$\sum_{r=0}^{r=n-1} P_r y^r.$$

We require then that, for all values of k and r from 0, to $n-1$

$$\overline{P}_r + r\overline{y_k} - \overline{\chi'(y_k)} < -1.$$

Now

$$\chi'(y_k)=(y_k-y_1)(y_k-y_2) \ldots (y_k-y_{k-1})(y_k-y_{k+1}) \ldots (y_k-y_n)$$

whence $$\overline{\chi'(y_k)}=\overline{y_1}+\overline{y_2}+ \ldots +\overline{y_{k-1}}+(n-k)\overline{y_k}$$

so that $$\overline{P}_r < -1+\overline{y_1}+ \ldots +\overline{y_{k-1}}+(n-k-r)\overline{y_k}$$

Now, to write $k+1$ for k is to change the right hand side of this inequality by $\overline{y_k}-(n-k-r)\overline{y_k}+(n-k-r-1)\overline{y_{k+1}}$; *i.e.*, by $(n-k-r-1)(\overline{y_{k+1}}-\overline{y_k})$.

This is negative if $k<n-r-1$

vanishes if $k=n-r-1$

is positive if $k>n-r-1$.

So there is a minimum value when $k=n-r-1$, and we must therefore have

$$\overline{\mathrm{P}_r}<-1+\overline{y_1}+\overline{y_2}+\ldots+\overline{y_{n-r-1}}.$$

Let $n-r-1=k_\alpha+\beta$ (and lie between k_α and $k_{\alpha+1}$),

then $$\overline{\mathrm{P}_r}<-1+n_1m_1+\ldots+n_\alpha m_\alpha+\beta\frac{m_{\alpha+1}}{\mu_{\alpha+1}}$$

Therefore

$$\overline{\mathrm{P}_r}\overline{\overline{<}}\mathrm{E}\left\{\sum_{i=1}^{i=\alpha}n_im_i+\beta\frac{m_{\alpha+1}}{\mu_{\alpha+1}}\right\}-1. \quad \text{(B)}$$

If this is to be true whatever r is, it must hold when we put $\alpha=0$;

wherefore $$\overline{\mathrm{P}_r}\overline{\overline{<}}\mathrm{E}\left(\beta\frac{m_1}{\mu_1}\right)-1\overline{\overline{<}}\mathrm{E}(\beta\sigma_1)-1$$

where r is one of the numbers $n-1$, $n-2$, ... $n-k_1$ and β is less than k_1: for $\beta=n-r-1$.

Now $\overline{\mathrm{P}_r}$ cannot be negative, therefore the smallest value assignable to β is the least which makes

$$\mathrm{E}(\beta\sigma_1)=1\text{ ; } i.e., \text{ is } (\beta'\equiv)\mathrm{E}\left(\frac{1}{\sigma}\right)+1.$$

We must then have $\mathrm{P}_{n-\beta'-1}\,y^{n-\beta'-1}$ *as the highest term in* $f_1(x, y)$.

This condition, necessary—and, as we see without difficulty, sufficient also; for the values assigned by equation (B) to $\overline{\mathrm{P}_r}$ are clearly positive when α is greater than zero—can always be satisfied unless $\beta'=n$.

This can only happen in two cases, viz.: when $\sigma=\frac{1}{n-1}$ or $\sigma=\frac{1}{n}$. In these two cases it can be easily shown that a *single* integral of the given form can be expressed by means of algebraic and logarithmic functions; so that ABEL'S theorem becomes unnecessary.

Except then in these two cases it is always possible by satisfying the conditions (A) and (B) to render the sum of the series of functions equal to a constant.*

The number of arbitrary constants, being equal to the number of relations connecting the variables of the functions which we sum, will by art. 20 (G) be

$$\sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n - \tfrac{1}{2}n + 1.$$

It is not necessary that we should assume $F_0(x)=1$ for the correctness of the processes of the last two pages.

Our equations will be the same if for any other reason $F_0(x)$ disappears from the general formula, and reduces it to the case of art. 9.

But this will happen if in the denominator of $\frac{1}{f_2(x)}\Sigma\frac{f_1(x,y)}{\chi'(y)}\log\theta y$ there is no factor also occurring in $F_0(x)$; and this will be so *if* $F_0(x)$ *and* $\frac{\chi'(y)}{f_1(x,y)}$ *do not vanish for any the same value of* x.

If this condition hold the results just arrived at will remain true.

APPENDIX.

LEMMA.

To find the values (i) *of the integral parts,* (ii) *of the fractional parts,* (iii) *of the complements to the fractional parts of the series of terms*

$$\frac{a}{n},\ \frac{a+b}{n},\ \frac{a+2b}{n},\ \ldots\ \frac{a+(n-1)b}{n}$$

where n is a positive integer, and a and b are integers positive or negative.

By the *integral part* of a term we mean the integer next less than or equal to it; by the *fractional part* that positive fraction (zero included) which added to the *integral part* gives the number; by the *complement of the fractional part* that fraction which added to the given number produces the next higher integer.

Let these functions of the term be denoted by the symbols $E\ \epsilon\ \epsilon'$.

* A notable particular case is that in which $f_1(x, y)$ consists of a single term, $x^k y^m$; where m is so chosen as to satisfy the condition (B) above, and k so as to satisfy the equation (i) of the last page.

Then, by the theory of numbers, if b and n are prime, the integers

$$n\epsilon\frac{a}{n},\ n\epsilon\frac{a+b}{n},\ \ldots\ n\epsilon\frac{a+\overline{n-1}.b}{n}$$

will be (in some order) the series 0, 1, 2 . . . , $n-1$; while if b and n are divisible by c, c being their greatest common measure, the integers

$$n\epsilon\frac{a}{n},\ n\epsilon\frac{a+b}{n},\ \ldots,\ n\epsilon\frac{a+\overline{n-1}.b}{n}$$

form an arithmetical progression whose common difference is c, repeated c times; and the smallest term of this progression is the remainder when c is divided into a.

If this remainder be called d we have

$$\sum_{l=0}^{l=n-1}\epsilon\frac{a+lb}{n}=\frac{c}{n}\left\{d+(d+c)+(d+2c)+\ldots \text{ to } \frac{n}{c} \text{ terms}\right\}$$

$$=d+\frac{n-c}{2}$$

whence

$$\sum_{l=0}^{l=n-1}\epsilon'\frac{a+lb}{n}=n-\sum_{l=0}^{l=n-1}\epsilon\frac{a+lb}{n}$$

$$=-d+\frac{n+c}{2}$$

and

$$\sum_{l=0}^{l=n-1}\mathrm{E}\frac{a+lb}{n}=\Sigma\frac{a+lb}{n}+d-\frac{n+c}{2}$$

Corollary i.

If c the greatest common factor of b and n also divides a, then $d=0$, and we have the simpler forms

$$\Sigma\epsilon'=\frac{n+c}{2},\ \Sigma\epsilon=\frac{n-c}{2}.$$

Corollary ii.

The sum of the fractional parts of any n *terms of the series* (*repetitions being allowed*) *differs from the sum of the fractional parts of the values of the same terms when* a *is put equal to zero, by an integer.*

For, if the sum of the coefficients of b in the numerators of the n terms be λ, then

$$\Sigma\epsilon_1 = a + \frac{\lambda b}{n} - \Sigma E_1 \text{ in the first case}$$

$$\Sigma\epsilon_2 = \frac{\lambda b}{n} - \Sigma E_2 \text{ in the second}$$

(the notation being obvious)

wherefore $$\Sigma\epsilon - \Sigma\epsilon' = a + \Sigma E_2 - \Sigma E_1 = \text{ an integer}$$

which is the required result.

List of Errata.

In Abel's Memoir the following slighter mistakes should be corrected :—

Page 184, ll. 12, 13,	for F	read T
192, l. 4,	for $\theta_1 x - \beta$	read $(x-\beta)^\nu$.
200, l. 3,	for hy^m	read $hy^{\mu'}$.
207, l. 9,	for χy	read $\chi' y$.
231, l. 2,	for z_1	read z_2.
231, l. 3,	for z_2	read z_3.
233,	for γ	read r throughout.
240,	for s^m	read s_m throughout.
243, l. 2,	for $n\delta_{2,\pi}$	read $n\delta_{2,\rho}$.
252, last line,	for s_{m-1}	read s_{n-1}.
255, last line but one,	for z	read 2.

There are besides these the inaccuracies referred to by M. Libri (the editor of the paper) as occurring on pp. 226–8.

These are too numerous to be treated otherwise than by re-writing the pages, which has therefore been done ; and they immediately follow.

" Alors l'equation (92) donnera les suivantes :—

$$f(12) = f(11) - \tfrac{4}{3} - A_0^{\mathrm{i}}, \quad \text{donc } A_0^{\mathrm{i}} = \tfrac{2}{3} \quad f(12) = f(11) - 2.$$
$$f(10) = f(11) + \tfrac{4}{3} - A_2^{\mathrm{i}}, \quad \text{donc } A_2^{\mathrm{i}} = \tfrac{1}{3} \quad f(10) = f(11) + 1.$$
$$f(9) = f(6) - \tfrac{3}{5} - A_3^{\mathrm{ii}}, \quad \text{donc } A_3^{\mathrm{ii}} = \tfrac{2}{5} \quad f(9) = f(6) - 1.$$
$$f(8) = f(6) - \tfrac{2}{5} - A_4^{\mathrm{ii}}, \quad \text{donc } A_4^{\mathrm{ii}} = \tfrac{3}{5} \quad f(8) = f(6) - 1.$$
$$f(7) = f(6) - \tfrac{1}{5} - A_5^{\mathrm{ii}}, \quad \text{donc } A_5^{\mathrm{ii}} = \tfrac{4}{5} \quad f(7) = f(6) - 1.$$
$$f(5) = f(6) + \tfrac{1}{5} - A_7^{\mathrm{ii}}, \quad \text{donc } A_7^{\mathrm{ii}} = \tfrac{1}{5} \quad f(5) = f(6).$$
$$f(3) = f(4) - \tfrac{1}{2} - A_9^{\mathrm{iii}}, \quad \text{donc } A_9^{\mathrm{iii}} = \tfrac{1}{2} \quad f(3) = f(4) - 1.$$
$$f(2) = f(4) - 1 - A_{10}^{\mathrm{iii}}, \quad \text{donc } A_{10}^{\mathrm{iii}} = 0 \quad f(2) = f(4) - 1.$$
$$f(1) = f(4) - \tfrac{3}{2} - A_{11}^{\mathrm{iii}}, \quad \text{donc } A_{11}^{\mathrm{iii}} = \tfrac{1}{2} \quad f(1) = f(4) - 2.$$

" Pour trouver maintenant $f(0)$, $f(4)$, $f(6)$, $f(11)$, il faut chercher les limites de θ_1, θ_2, θ_3, θ_4.

" Or les équations (103), qui déterminent ces limites donnent

$$\theta_1 > \frac{11-\alpha_1}{5} - \frac{3A_\beta^{\text{i}}}{17}\ ;\ \text{d'où}\ \theta_1 > -\frac{1}{5}-\frac{2}{17}\ ;\ 0\ ;\ \frac{1}{5}-\frac{1}{17}.$$

$$\theta_1 < \frac{11-\alpha_1}{5} + \frac{3A_\beta^{\text{ii}}}{17}\ ;\ \text{d'où}\ \theta_1 < \frac{2}{5}+\frac{6}{5.17}\ ;\ \frac{3}{5}+\frac{9}{5.17}\ ;\ \frac{4}{5}+\frac{12}{5.17}\ ;\ 1\ ;\ \frac{6}{5}+\frac{3}{5.17}.$$

" Il suit de là que

$$\theta_1 > \frac{12}{85} < \frac{40}{85}.$$

" On a aussi

$$\theta_2 > \frac{6-\alpha_2}{2} - \frac{5A_\beta^{\text{ii}}}{7}\ ;\ \text{d'où}\ \theta_2 > \ldots,\ \frac{1}{2}-\frac{1}{7}$$

$$\theta_2 < \frac{6-\alpha_3}{2} + \frac{5A_\beta^{\text{iii}}}{2}\ ;\ \text{d'où}\ \theta_2 < \frac{2}{2},\ \frac{3}{2}+\frac{5}{14},\ \ldots$$

" Il suit que

$$\theta_2 > \frac{5}{14} < 1$$

$$\theta_3 > \frac{4-\alpha_3}{4} - \frac{A_\beta^{\text{iii}}}{2}\ ;\ \text{d'où}\ \theta_3 > 0,\ \frac{1}{4}-\frac{1}{4},\ \frac{2}{4},\ \frac{3}{4}-\frac{1}{4}$$

$$\theta_3 < \frac{4-\alpha_4}{4} + \frac{A_\beta^{\text{iv}}}{2}\ ;\ \text{d'où}\ \theta_3 < 1$$

" Il suit que

$$\theta_3 > \tfrac{1}{2} < 1.$$

" Maintenant l'équation (97) donne

$$f(\rho_m) - f(\rho_{m-1}) > (\rho_{m-1} - \rho_m)(\theta''_{m-1}\sigma_{m-1} + \overline{1-\theta''_{m-1}}\sigma_m)$$
$$f(\rho_m) - f(\rho_{m-1}) < (\rho_{m-1} - \rho_m)(\theta'_{m-1}\sigma_{m-1} + \overline{1-\theta'_{m-1}}\sigma_m)$$

où θ''_{m-1} est la plus petite, et θ'_{m-1} la plus grande valeur de θ_{m-1} ; donc on trouvera, en faisant $m=2, 3, 4$,

$$f(6)-f(11) > 5(\tfrac{12}{85}\cdot\tfrac{4}{3} + \overline{1-\tfrac{12}{85}}\cdot\tfrac{1}{5}\ );\ (=1+\tfrac{68}{85})$$
$$f(6)-f(11) < 5(\tfrac{40}{85}\cdot\tfrac{4}{3} + \overline{1-\tfrac{40}{85}}\cdot\tfrac{1}{5}\ );\ (=3+\tfrac{34}{51})$$
$$f(4)-f(6) > 2(\tfrac{5}{14}\cdot\tfrac{1}{5} - \overline{1-\tfrac{5}{14}}\cdot\tfrac{1}{2}\ );\ (=-\tfrac{1}{2})$$
$$f(4)-f(6) < 2(1\cdot\tfrac{1}{5} - \overline{1-1}\tfrac{1}{2}\ );\ (=\tfrac{2}{5})$$
$$f(0)-f(4) > 4(\tfrac{1}{2}\cdot\overline{-\tfrac{1}{2}} + \overline{1-\tfrac{1}{2}}\cdot\overline{-1});\ (=-3)$$
$$f(0)-f(4) < 4(1\overline{-\tfrac{1}{2}} + \overline{1-1}\cdot\overline{-1});\ (=-2)$$

donc on aura pour $f(6)-f(11), f(4)-f(6), f(0)-f(4)$ les valeurs suivantes :—

$$f(6)-f(11)=2,\ 3. \quad f(4)-f(6)=0. \quad f(0)-f(4)=-3,\ -2.$$

d'où

$$\begin{aligned}
&f(11)=f(6)-2, f(6)-3\ ;\ f(4)=f(6);\ f(0)=f(6)-3, f(6)-2\\
&f(12)=f(6)-4, f(6)-5\ ;\ f(10)=f(6)-1, f(6)-2\\
&f(9)=f(6)-1\ ;\ f(8)=f(6)-1\ ;\ f(7)=f(6)-1, f(5)=f(6)\\
&f(3)=f(6)-1\ ;\ f(2)=f(6)-1\ ;\ f(1)=f(6)-2
\end{aligned}$$

"En exprimant donc toutes ces quantités par $f(12)$ on voit que les fonctions $q_{12}, q_{11}, \ldots q_0$ sont respectivement des degrés suivants :—

(12)	(11)	(10)	(9)	(8)	(7)	(6)	(5)	(4)	(3)	(2)	(1)	(0)
θ,	$\theta+2$,	$\theta+3$,	$\theta+3$,	$\theta+3$,	$\theta+3$,	$\theta+4$,	$\theta+4$,	$\theta+4$,	$\theta+3$,	$\theta+3$,	$\theta+2$,	$(\theta+2,\ \theta+1)$

ou

(12)	(11)	(10)	(9)	(8)	(7)	(6)	(5)	(4)	(3)	(2)	(1)	(0)
θ,	$\theta+2$,	$\theta+3$,	$\theta+4$,	$\theta+4$,	$\theta+4$,	$\theta+5$,	$\theta+5$,	$\theta+5$,	$\theta+4$,	$\theta+4$,	$\theta+3$,	$(\theta+3,\ \theta+2)$

où θ est le degré de la fonction q_{12}.

"De là suit que

$$\begin{aligned}\alpha=f(0)+f(1)+\ \ldots\ +f(12)+12=13\theta+47,\ 13\theta+48\\ 13\theta+57,\ 13\theta+58\end{aligned}$$

et

$$\begin{aligned}\mu&=n'\mu'\left(f(\rho_1)+\rho_1\frac{m'}{\mu'}\right)+\ \ldots\ +\ \ldots\ +\ \ldots\\ &=3f(11)+44+5f(6)+6+4f(4)-8+f(0)\end{aligned}$$

c'est à dire

$$\begin{aligned}\mu=13\theta+95,\ \ 13\theta+96\\ 13\theta+105,\ 13\theta+106\end{aligned}$$

"La valeur de $\mu-\alpha$ deviendra donc

$$\mu-\alpha=38$$

comme nous avons trouvé plus haut."

Addition to Mr. ROWE'S *Memoir.*

By Professor CAYLEY, *F.R.S.*

Received May 27,—Read June 10, 1880.

IN ABEL'S general theorem y is an irrational function of x determined by an equation $\chi(y)=0$ (or say $\chi(x, y)=0$) of the order n as regards y: and it was shown by him that the sum of any number of the integrals considered may be reduced to a sum of γ integrals; where γ is a determinate number depending only on the form of the equation $\chi(x, y)=0$, and given in his equation (62) p. 206: viz., if (solving the equation so as to obtain from it developments of y in descending series of powers of x) we have*

$$\begin{array}{llll} n_1\mu_1 & \text{series each of the form} & & y=x^{\frac{m_1}{\mu_1}}+\ldots, \\ n_2\mu_2 & \quad ,, & ,, & y=x^{\frac{m_2}{\mu_2}}+\ldots, \\ \vdots & & & \vdots \\ n_k\mu_k & \quad ,, & ,, & y=x^{\frac{m_k}{\mu_k}}+\ldots, \end{array}$$

(so that $n=n_1\mu_1+n_2\mu_2\ldots+n_k\mu_k$), then γ is a determinate function of n_1, m_1, μ_1; n_2, m_2, μ_2; $\ldots$ n_k, m_k, μ_k.

Mr. ROWE has expressed ABEL'S γ in the following form, viz., assuming

$$\frac{m_1}{\mu_1}>\frac{m_2}{\mu_2}\ldots>\frac{m_k}{\mu_k},$$

* The several powers of x have coefficients: the form really is $y=A_1x^{\frac{m_1}{\mu_1}}+\ldots$, which is regarded as representing the μ_1 different values of y obtained by giving to the radical $x^{\frac{1}{\mu_1}}$ each of its μ_1 values, and the corresponding values to the radicals which enter into the coefficients of the series: and (so understanding it) the meaning is that there are n_1 such series each representing μ_1 values of y. It is assumed that the series contains *only* the radical $x^{\frac{1}{\mu_1}}$, that is, the indices after the leading index $\frac{m_1}{\mu_1}$ are $\frac{m_1-1}{\mu_1}$, $\frac{m_1-2}{\mu_1}$, ...; a series such as $y=A_1x^{\frac{4}{3}}+B_1x^{\frac{3}{5}}+\ldots$, depending on the two radicals $x^{\frac{1}{3}}$, $x^{\frac{1}{5}}$ represents 15 different values, and would be written $y=A_1x^{\frac{20}{15}}+\ldots$, or the values of m_1 and μ_1 would be 20 and 15 respectively: in a case like this where $\frac{m_1}{\mu_1}$ is not in its least terms, the number of values of the leading coefficient A_1 is equal, not to μ_1, but to a submultiple of μ_1. But the case is excluded by ABEL'S assumption that $\frac{m_1}{\mu_1}$, $\frac{m_2}{\mu_2}$ ..., are fractions each of them in its least terms.

then this expression is

$$\gamma = \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n - \tfrac{1}{2}n + 1,$$

or what is the same thing, for n writing its value $\Sigma n\mu$,

$$\gamma = \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n\mu - \tfrac{1}{2}\Sigma n + 1,$$

where in the first sum r, s have each of them the values 1, 2, . . . k, subject to the condition $s>r$; in each of the other sums n, m, and μ are considered as having the suffix r, which has the values 1, 2, . . . k.

It is a leading result in RIEMANN'S theory of the Abelian integrals that γ is the deficiency (Geschlecht) of the curve represented by the equation $\chi(x, y)=0$: and it must consequently be demonstrable *à posteriori* that the foregoing expression for γ is in fact = deficiency of curve $\chi(x, y)=0$. I propose to verify this by means of the formulæ given in my paper "On the Higher Singularities of a Plane Curve," Quart. Math. Jour., vol. vii., pp. (1866) 212–222.

It is necessary to distinguish between the values of $\frac{m}{\mu}$ which are $>$, $=$, and $<$ 1; and to fix the ideas I assume $k=7$, and

$$\frac{m_1}{\mu_1}, \frac{m_2}{\mu_2}, \frac{m_3}{\mu_3} \text{ each } > 1,$$

$$\frac{m_4}{\mu_4}=1; \text{ say } m_4=\mu_4=\lambda; \text{ and } n_4=\theta,$$

$$\frac{m_5}{\mu_5}, \frac{m_6}{\mu_6}, \frac{m_7}{\mu_7} \text{ each } < 1,$$

but it will be easily seen that the reasoning is quite general. I use Σ' to denote a sum in regard to the first set of suffixes 1, 2, 3, and Σ'' to denote a sum in regard to the second set of suffixes 5, 6, 7. The foregoing value of n is thus

$$n = \Sigma' n\mu + \lambda\theta + \Sigma'' n\mu.$$

Introducing a third coordinate z for homogeneity, the equation $\chi(x, y)=0$ of the curve will be

$$0 = \left(yz^{\frac{m_1}{\mu_1}-1} - x^{\frac{m_1}{\mu_1}}\right)^{n_1\mu_1} \ldots \left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta} \left(y - x^{\frac{m_5}{\mu_5}} z^{1-\frac{m_5}{\mu}}\right) \ldots$$

where it is to be observed that $(\quad)^{n_1\mu_1}$ is written to denote the product of $n_1\mu_1$ different series each of the form $yz^{\frac{m_1}{\mu_1}-1} - A_1 x^{\frac{m_1}{\mu_1}} \ldots$; these divide themselves into n_1 groups, each a product of μ_1 series; and in each such product the μ_1 coefficients A_1 are in general the

μ_1 values of a function containing a radical $a^{\frac{1}{\mu_1}}$ and are thus different from each other: it is in what follows in effect assumed not only that this is so, but that all the $n_1\mu_1$ coefficients A_1 are different from each other:* the like remarks apply to the other factors. It applies in particular to the term $\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$, viz., it is assumed that the coefficients A in the $\lambda\theta$ series $y=Ax^{\frac{\lambda}{\lambda}}+\ldots$, are all of them different from each other. These assumptions as to the leading coefficients really imply ABEL's assumption that $\frac{m_1}{\mu_1}, \ldots \frac{m_k}{\mu_k}$ are all of them fractions in their least terms, and in particular that $\frac{\lambda}{\lambda}$ is a fraction in its least terms, viz., that $\lambda=1$: I retain however for convenience the general value λ, putting it ultimately $=1$.

In the product of the several infinite series the terms containing negative powers all disappear of themselves; and the product is a rational and integral function $F(x, y, z)$ of the coordinates, which on putting therein $z=1$ becomes $=\chi(x, y)$. The equation of the curve thus is $F(x, y, z)=0$; and the order is $=\frac{m_1}{\mu_1}n_1\mu_1+\ldots+\lambda\theta+n_5\mu_5+\ldots$, $=m_1n_1+\ldots+\lambda\theta+n_5\mu_5+\ldots$; viz., if K is the order of the curve $\chi(x, y)=0$, then $K=\Sigma' nm+\lambda\theta+\Sigma'' n\mu$.

The curve has singularities (singular points) at infinity, that is, on the line $z=0$: viz.—

First, a singularity at $(z=0, x=0)$, where the tangent is $x=0$, and which (writing for convenience $y=1$) is denoted by the function

$$\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}\ldots;$$

where observe that the expressed factor indicates n_1 branches $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{m_1-\mu_1}$, or say $n_1(m_1-\mu_1)$ partial branches $z-x^{\frac{m_1}{m_1-\mu_1}}$, that is $n_1(m_1-\mu_1)$ partial branches $z=A_1x^{\frac{m_1}{m_1-\mu_1}}+\ldots$, with in all $n_1(m_1-\mu_1)$ distinct values of A_1; and the like as regards the unexpressed factors with the suffixes 2 and 3.

Secondly, a singularity at $(z=0, y=0)$, where the tangent is $y=0$, and which (writing for convenience $x=1$) is denoted by the function

$$\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}\ldots;$$

where observe that the expressed factor indicates n_5 branches $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{\mu_5-m}$, or say

* This assumption is virtually made by ABEL, p. 198, in the expression "alors on aura en général, excepté quelques cas particuliers que je me dispense de considérer: $h(y'-y'')=hy'$, &c.": viz., the meaning is that the degree of y' being greater than or equal to that of y'', then the degree of $y'-y''$ is equal to that of y'': of course when the degrees are equal, this implies that the coefficients of the two leading terms must be unequal.

$n_5(\mu_5-m_5)$ partial branches $z-y^{\frac{\mu_5}{\mu_5-m_5}}$, that is $n_5(\mu_5-m_5)$ partial branches $z=A_5y^{\frac{\mu_5}{\mu_5-m_5}}+\ldots$, with in all $n_5(\mu_5-m_5)$ distinct values of A_5: and the like as regards the unexpressed factors with the suffixes 6 and 7.

Thirdly, singularities at the θ points $(z=0,\ y-Ax=0)$, A having here θ distinct values, at any one of which the tangent is $y-Ax=0$, and which are denoted by the function

$$\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}:$$

but in the case ultimately considered λ is $=1$; and these are then the θ ordinary points at infinity, $(z=0,\ y-Ax=0)$.

According to the theory explained in my paper above referred to, these several singularities are together equivalent to a certain number $\delta'+\kappa'$ of nodes and cusps, viz., we have

$$\delta'=\tfrac{1}{2}M-\tfrac{3}{2}\Sigma(\alpha-1)$$
$$\kappa'=\qquad\Sigma(\alpha-1),$$

hence

$$\delta'+\kappa'=\tfrac{1}{2}M-\tfrac{1}{2}\Sigma(\alpha-1)$$

and (assuming that there are no other singularities) the deficiency

$$\tfrac{1}{2}(K-1)(K-2)-\delta'-\kappa'$$

is

$$=\tfrac{1}{2}(K-1)(K-2)-\tfrac{1}{2}M+\tfrac{1}{2}\Sigma(\alpha-1)$$

this should be equal to the before-mentioned value of γ, viz., we ought to have

$$(K-1)(K-2)-M+\Sigma(\alpha-1)=2\sum_{s>r}n_rm_rn_s\mu_s+\Sigma n^2m\mu-\Sigma nm-\Sigma n\mu-\Sigma n+2$$

or, as it will be convenient to write it,

$$M=K^2-3K+\Sigma(\alpha-1)-2\sum_{s>r}n_rm_rn_s\mu_s-\Sigma n^2m\mu+\Sigma nm+\Sigma n\mu+\Sigma n,$$

which is the equation which ought to be satisfied by the values of M and $\Sigma(\alpha-1)$ calculated, according to the method of my paper, for the foregoing singularities of the curve.

We have as before

$$K=\Sigma'nm+\Sigma''n\mu+\theta\lambda.$$

The term $\sum_{s>r}n_rm_rn_s\mu_s$, written at length, is

$$\begin{aligned}
= \quad & n_1m_1(n_2\mu_2+n_3\mu_3+\theta\lambda+n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & n_2m_2(\qquad n_3\mu_3+\theta\lambda+n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & n_3m_3(\qquad \theta\lambda+n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & \ \theta\lambda\ (\qquad n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & n_5m_5(\qquad n_6\mu_6+n_7\mu_7)\\
+ & n_6m_6(\qquad n_7\mu_7),
\end{aligned}$$

which is

$$=\underset{s>r}{\Sigma'}n_rm_rn_s\mu_s+\theta\lambda(\Sigma'nm+\Sigma''n\mu)+\Sigma'nm.\Sigma''n\mu+\underset{s>r}{\Sigma''}n_rm_rn_s\mu_s.$$

We have moreover

$$\begin{aligned}
\Sigma n^2m\mu &=\Sigma'n^2m\mu+\theta^2\lambda^2+\Sigma''n^2m\mu,\\
\Sigma nm &=\Sigma'nm \quad +\theta\lambda \ +\Sigma''nm,\\
\Sigma n\mu &=\Sigma'n\mu \quad +\theta\lambda \ +\Sigma''n\mu,\\
\Sigma n &=\Sigma'n \quad +\theta \ +\Sigma''n.
\end{aligned}$$

We next calculate $\Sigma(\alpha-1)$.

For the singularity

$$\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}\ldots$$

each branch $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{m_1-\mu_1}$ gives $\alpha=m_1-\mu_1$, and the value of $\Sigma(\alpha-1)$ for this singularity is $n_1(m_1-\mu_1-1)+n_2(m_2-\mu_2-1)+n_3(m_3-\mu_3-1)$, which is

$$=\Sigma'nm-\Sigma'n\mu-\Sigma'n.$$

For the singularity

$$\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}\ldots$$

each branch $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{\mu_5-m_5}$ gives $\alpha=\mu_5-m_5$, and the value of $\Sigma(\alpha-1)$ for this singularity is $n_5(\mu_5-m_5-1)+n_6(\mu_6-m_6-1)+n_7(\mu_7-m_7-1)$, which is

$$=\Sigma''n\mu-\Sigma''nm-\Sigma''n.$$

For each of the θ singularities

$$\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$$

we have $\alpha=\lambda$ and the value of $\Sigma(\alpha-1)$ is $=\theta(\lambda-1)$: this is $=0$ for the value $\lambda=1$, which is ultimately attributed to λ.

The complete value of $\Sigma(\alpha-1)$ is thus

$$=\Sigma' nm-\Sigma'' nm-\Sigma' n\mu+\Sigma'' n\mu-\Sigma' n-\Sigma'' n-\theta\lambda-\theta.$$

Substituting all these values we have

$$\begin{aligned}\mathrm{M}=&(\Sigma' nm+\Sigma'' n\mu)^2+2\theta\lambda(\Sigma' nm+\Sigma'' n\mu)+(\theta\lambda)^2\\ &-3(\Sigma' nm+\Sigma'' n\mu)-3\theta\lambda\\ &+\Sigma' nm-\Sigma'' nm-\Sigma' n\mu+\Sigma'' n\mu-\Sigma' n-\Sigma'' n+\theta\lambda-\theta\\ &-2\sum_{s>r} n_r m_r n_s \mu_s-2\theta\lambda(\Sigma' nm+\Sigma'' n\mu)-2\Sigma' nm.\Sigma'' n\mu-2\sum_{s>r}{}'' n_r m_r n_s \mu_s\\ &-\Sigma' n^2 m\mu-(\theta\lambda)^2-\Sigma'' n^2 m\mu\\ &+\Sigma' nm+\theta\lambda+\Sigma'' nm\\ &+\Sigma' n\mu+\theta\lambda+\Sigma'' n\mu\\ &+\Sigma' n+\theta+\Sigma'' n,\end{aligned}$$

or reducing

$$\begin{aligned}\mathrm{M}=&(\Sigma' nm)^2-\Sigma' nm-\Sigma' n^2 m\mu-2\sum_{s>r}{}' n_r m_r n_s \mu_s\\ &+(\Sigma'' n\mu)^2-\Sigma'' n\mu-\Sigma'' n^2 m\mu-2\sum_{s>r}{}'' n_r m_r n_s \mu_s\,;\end{aligned}$$

and it is to be shown that the two lines of this expression are in fact the values of M belonging to the singularities $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}$ . . . , and $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}$ . . . respectively. We assume $\lambda=1$, and there is thus no singularity $\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$.

I recall that, considering the several partial branches which meet at a singular point, M denotes the sum of the number of the intersections of each partial branch by every other partial branch (so that for each pair of partial branches the intersections are to be counted *twice*). Supposing that the tangent is $x=0$, and that for any two branches we have $z_1=\mathrm{A}_1x^{p_1}$, $z_2=\mathrm{A}_2x^{p_2}$ (where p_1, p_2 are each equal to or greater than 1), then if $p_2=p_1$, and $z_1-z_2=(\mathrm{A}_1-\mathrm{A}_2)x^{p_1}$ where $\mathrm{A}_1-\mathrm{A}_2$ not$=0$ (an assumption which has been already made as regards the cases about to be considered), then the number of intersections is taken to be $=p_1$; and if p_1 and p_2 are unequal, then *taking* p_2 *to be the greater of them*, the leading term of z_1-z_2 is $=\mathrm{A}_1x^{p_1}$, and the number of intersections is taken to be $=p_1$; viz., in the case of unequal exponents, it is equal to the smaller exponent.

Consider now the singularity $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}$ . . . ; and first the intersections of a partial branch $z-x^{\frac{m_1}{m_1-\mu_1}}$ by each of the remaining $n_1(m_1-\mu_1)-1$ partial branches of the same set: the number of intersections with any one of these is $=\frac{m_1}{m_1-\mu_1}$; and consequently the number with all of them is $=\frac{m_1}{m_1-\mu_1}[n_1(m_1-\mu_1)-1]$. But we obtain this same number from each of the $n_1(m_1-\mu_1)$ partial branches, and thus the whole number is $n_1(m_1-\mu_1)\frac{m_1}{m_1-\mu_1}[n_1(m_1-\mu_1)-1], =n_1m_1[n_1(m_1-\mu_1)-1]$.

Taking account of the other sets, each with itself, the whole number of such intersections is

$$n_1m_1[n_1(m_1-\mu_1)-1]+n_2m_2[n_2(m_2-\mu_2)-1]+n_3m_3[n_3(m_3-\mu_3)-1],$$

which is

$$=\Sigma' n^2m^2-\Sigma' n^2m\mu-\Sigma' nm.$$

Observe now that $\frac{m_1}{\mu_1}>\frac{m_2}{\mu_2}$, that is $\frac{\mu_1}{m_1}<\frac{\mu_2}{m_2}$, and that, these being each <1, we thence have $1-\frac{\mu_1}{m_1}>1-\frac{\mu_2}{m_2}$, that is $\frac{m_1-\mu_1}{m_1}>\frac{m_2-\mu_2}{m_2}$: and we thus have

$$\frac{m_1}{m_1-\mu_1}<\frac{m_2}{m_2-\mu_2}<\frac{m_3}{m_3-\mu_3}.$$

Considering now the intersections of partial branches of the two sets $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}$ and $\left(z-x^{\frac{m_2}{m_2-\mu_2}}\right)^{n_2(m_2-\mu_2)}$ respectively, a partial branch $z-x^{\frac{m_1}{m_1-\mu_1}}$ gives with each partial branch of the other set a number $=\frac{m_1}{m_1-\mu_1}$; and in this way taking each partial branch of each set, the number is $n_1(m_1-\mu_1).n_2(m_2-\mu_2).\frac{m_1}{m_1-\mu_1}, =n_1m_1n_2(m_2-\mu_2)$; and thus for all the sets the number is

$$=n_1m_1n_2(m_2-\mu_2)+n_1m_1n_3(m_3-\mu_3)+n_2m_2n_3(m_3-\mu_3),$$

which is

$$=\Sigma' n_rm_rn_sm_s-\sum_{s>r}{}' n_rm_rn_s\mu_s,$$

where in the first sum the Σ' refers to each pair of values of the suffixes. But the intersections are to be taken twice; the number thus is

$$=2\Sigma' n_rm_rn_sm_s-2\sum_{s>r} n_rm_rn_s\mu_s.$$

Adding the foregoing number

$$\Sigma' n^2m^2-\Sigma' n^2m\mu-\Sigma' nm,$$

the whole number for the singularity in question is

$$=(\Sigma' nm)^2-\Sigma' nm-\Sigma' n^2m\mu-2\sum_{s>r}{}' n_rm_rn_s\mu_s.$$

Similarly for the singularity $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}\ldots$; taking each set with itself, the number of intersections is

$$n_5\mu_5[n_5(\mu_5-m_5)-1]+n_6\mu_6[n_6(\mu_6-m_6)-1]+n_7\mu_7[n_7(\mu_7-m_7)-1],$$

which is

$$=\Sigma''n^2\mu^2-\Sigma''n^2m\mu-\Sigma''n\mu.$$

We have here $\frac{m_5}{\mu_5}>\frac{m_6}{\mu_6}$ and each of these being less than 1, we have $1-\frac{m_5}{\mu_5}<1-\frac{m_6}{\mu_6}$, that is $\frac{\mu_5-m_5}{\mu_5}<\frac{\mu_6-m_6}{\mu_6}$, or $\frac{\mu_5}{\mu_5-m_5}>\frac{\mu_6}{\mu_6-m_6}$; and so

$$\frac{\mu_7}{\mu_7-m_7}<\frac{\mu_6}{\mu_6-m_6}<\frac{\mu_5}{\mu_5-m_5}.$$

Hence considering the two sets $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}$ and $\left(z-y^{\frac{\mu_6}{\mu_6-m_6}}\right)^{n_6(\mu_6-m_6)}$, a partial branch of the first set gives with a partial branch of the second set $\frac{\mu_6}{\mu_6-m_6}$ intersections: and the number thus obtained is $n_5(\mu_5-m_5).n_6(\mu_6-m_6).\frac{\mu_6}{\mu_6-m_6}$, $=n_5n_6\mu_6(\mu_5-m_5)$.

For all the sets the number is

$$n_5n_6\mu_6(\mu_5-m_5)+n_5n_7\mu_7(\mu_5-m_5)+n_6n_7\mu_7(\mu_6-m_6)$$

or taking this twice, the number is

$$=2\Sigma''n_r\mu_rn_s\mu_s-2\Sigma''_{s>r}n_rm_rn_s\mu_s$$

where in the first sum the Σ'' refers to each pair of suffixes. Adding the foregoing value

$$\Sigma''n^2\mu^2-\Sigma''n^2m\mu-\Sigma''n\mu,$$

the whole number for the singularity in question is

$$=(\Sigma''n\mu)^2-\Sigma''n\mu-\Sigma''n^2m\mu-2\Sigma''_{s>r}n_rm_rn_s\mu_s;$$

and the proof is thus completed.

Referring to the foot-note ante (p. 753), I remark that the theorem $\gamma=$ deficiency, is absolute, and applies to a curve with any singularities whatever: in a curve which has singularities not taken account of in ABEL's theory, the "quelques cas particuliers que je me dispense de considérer," the singularities not taken account of give rise to a diminution in the deficiency of the curve, and also to an equal diminution of the value of γ as determined by ABEL's formula; and the actual deficiency will be $=$ ABEL's γ $-$ such diminution, that is, it will be $=$ true value of γ.